



Introduction, Growth Rates and Index Numbers

Statistics I · Lecture 1

Today

- Course information
- Growth rates: simple, accumulated and average
- Index numbers: definition, base period, simple index

Course Introduction

Core Information

Instructor: Paulo Fagandini

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Main source for course material: Moodle@ISCAL

Official communication channel:  Institutional Email

Bibliography

Topic 1:

 Gancho Custódio, S. et al. (2022) Números Índices, Edições Sílabo

Topic 2:

 Murteira, B.; Silva Ribeiro, C.; Andrade e Silva, J. & Pimenta, C.,
Introdução à Estatística, Escolar Editora, McGraw-Hill, 2010

Topic 3:

 Ferreira, T., Custódio, S.G. (2023) Modelos Probabilísticos, Edições
Sílabo

Bibliography (EN)

Topic 1:

 The making of index numbers. 1st Ed 1922. Irving Fisher.

 A Practical Introduction to Index Numbers. 1st Ed 2015. Jeff Ralph, Rob O'Neill, Joe Winton.

Topics 2 and 3:

 Statistics for Business and Economics Global Edition. 10th Ed 2022. Paul Newbold, William Carlson, and Betty Thorne.

This last book is a bit expensive, but it can be rented for less money [here](#).

Continuous Assessment

Assessment Element	Weight (%)	Duration	Syllabus	Date
Midterm 1	60%	80 minutes	Topics 1 and 2	Nov 5 th
Midterm 2	40%	80 minutes	Topic 3	Dec 10 th

Each test has a minimum grade of 7.00 (out of 20), not 6.9, not 6.95.

Comprehensive Exam

Students may opt, on the day of the Exam, to take an Exam worth 100% of the grade. This Exam will cover Topics 1, 2, and 3.

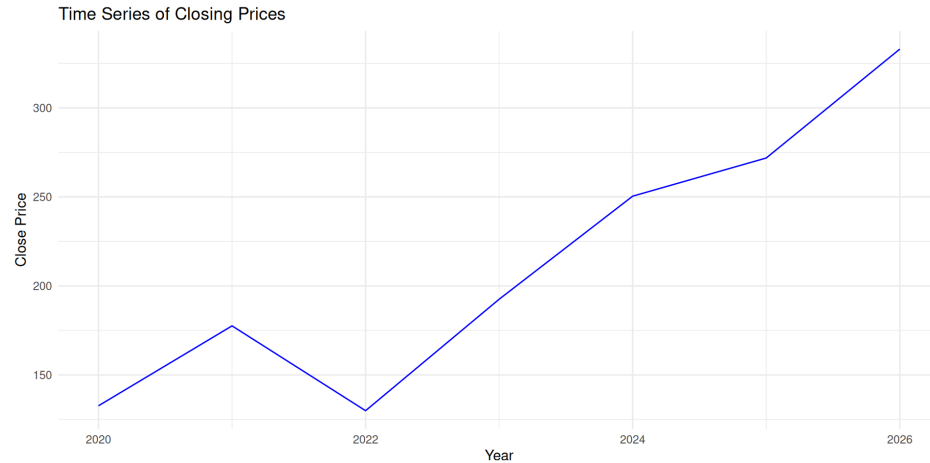
If you score less than 7.00 in the Midterm, or if you cannot attend the midterm for whatever reason, you **will** have to do the Comprehensive Exam.

Disclaimer

These slides are a free translation and adaptation from the slide deck for Estatística I by Prof. Sandra Custódio and Prof. Teresa Ferreira from the Lisbon Accounting and Business School, Polytechnic University of Lisbon.

Growth Rates

Example: Apple Stock



Source: Yahoo Finance

Date	Close
2020	132.69
2021	177.57
2022	129.93
2023	192.53
2024	250.42
2025	271.86
2026	333.02

Example: Apple Stock

Let the close price for Apple be represented by the variable y_t , so y_{2020} is the close price we got for 2020 (*i.e.* 132.69).

How much did y grow between 2020 and 2021?

$$\begin{aligned}\Delta y_{2021} &= y_{2021} - y_{2020} = 177.57 - 132.69 \\ &= 44.88\end{aligned}$$

Or we could say

$$\begin{aligned}y_{2021} &= y_{2020} + \Delta y_{2021} \\ 177.57 &= 132.69 + 44.88\end{aligned}$$

Example: Apple Stock

$$y_{2021} = y_{2020} + \Delta y_{2021}$$

$$y_{2021} = y_{2020} \left(1 + \frac{\Delta y_{2021}}{y_{2020}} \right)$$

$$y_{2021} = y_{2020} (1 + \delta_{2021}^y)$$

This, δ_{2021}^y , is the **growth rate** of y at year 2021.

Growth Rate

Growth Rate

Let y_t be a variable that might take different values over time. The **growth rate** at t is given by δ_t^y , and takes a value such that:

$$y_t = y_{t-1} (1 + \delta_t)$$

And then

$$\delta_t = \frac{y_t - y_{t-1}}{y_{t-1}}$$

Note: I dropped y from the notation in the formula because it is obvious that we are talking about the variable y .

Growth Rate

You can write the growth rate as a decimal or as a percentage. If you use decimal notation, consider at least 4 places, if you use percentage you use at least 2. Example: 0.0123 or 1.23%.

Interpretation: What is the percentage change for the variable y between $t - 1$ and t .

Example

$$177.57 = 132.69 + (177.57 - 132.69)$$

$$177.57 = 132.69 \left(1 + \frac{44.88}{132.69} \right)$$

$$177.57 = 132.69 (1 + 0.3382)$$

In this case $\delta_{2021}^y = 0.3382 = 33.82\%$.

Interpretation: Between 2020 and 2021, the price for Apple *increased* 33.82%.

Growth Rates

More generally

$$\delta_{t+k|t} = \frac{y_{t+k} - y_t}{y_t}$$

Or

$$y_{t+k} = y_t (1 + \delta_{t+k|t})$$

(homework: Show step by step how you go from one to the other.)

Note: In this case $\delta_{t+k|t}$ is what percentage the variable changed from t to $t + k$, and k is the number of periods.

Example

We had $y_{2020} = 132.69$, and $y_{2024} = 250.42$. We will try to find $\delta_{2024|2020}^y$

- What is the value of k in this case?

$$k = 2024 - 2020 = 4$$

- What is the growth rate between 2020 and 2024?

$$\delta_{2024|2020}^y = \frac{250.42 - 132.69}{132.69} = 0.8873 = 88.73\%$$

For Apple, the stock price grew 88.73% between 2020 and 2024, or the stock price for Apple in 2024 is 88.73% larger than in 2020.

Growth Rates

Warning

You should be careful with your interpretation and use of this *cumulative* growth rate, because it **does not** mean that y grew $\delta_{t+k|t}^y$ **per period** between t and $t + k$.

Growth Rates

Average Growth Rate

Let y_t be a variable that might take different values over time. The **average growth rate** between t and $t + k$ is given by $\bar{\delta}_{t+k|t}^y$, and takes a value such that:

$$y_{t+k} = y_t \left(1 + \bar{\delta}_{t+k|t}^y\right)^k$$

We will use something called the *geometric mean*.

I dropped y from the notation in the formula because it is obvious that we are talking about the variable y .

Growth Rates

Let's start with $y_t \dots$ and the traditional growth rates δ_t :

$$y_{t+1} = y_t(1 + \delta_{t+1})$$

$$y_{t+2} = y_{t+1}(1 + \delta_{t+2})$$

$$y_{t+3} = y_{t+2}(1 + \delta_{t+3})$$

Can we write y_{t+2} as a function of y_t ?

$$y_{t+2} = y_{t+1}(1 + \delta_{t+2}) = y_t(1 + \delta_{t+1})(1 + \delta_{t+2})$$

And y_{t+3} ?

$$y_{t+3} = y_t(1 + \delta_{t+1})(1 + \delta_{t+2})(1 + \delta_{t+3})$$

Growth Rates

Generalizing

$$y_{t+k} = y_t (1 + \delta_{t+1}) \dots (1 + \delta_{t+k-1}) (1 + \delta_{t+k})$$

When computing the average growth rate, we are trying to find $\bar{\delta}$ such that if the growth rate was the same every period, it would have taken y_t to the value of y_{t+k} all the same:

$$y_{t+k} = y_t (1 + \bar{\delta})(1 + \bar{\delta}) \dots (1 + \bar{\delta})$$

How many times is $(1 + \bar{\delta})$ multiplied in the expression?

Growth Rates

$$y_{t+k} = y_t \left(1 + \bar{\delta}\right)^k$$

$$\frac{y_{t+k}}{y_t} = \left(1 + \bar{\delta}\right)^k$$

$$\left(\frac{y_{t+k}}{y_t}\right)^{1/k} = \left(1 + \bar{\delta}\right)$$

$$\left(\frac{y_{t+k}}{y_t}\right)^{1/k} - 1 = \bar{\delta}$$

$$\bar{\delta}_{t+k|t} = \left(\frac{y_{t+k}}{y_t}\right)^{1/k} - 1$$

Example

We had

$$\delta_{2024|2020}^y = \frac{250.42 - 132.69}{132.69} = 0.8873 = 88.73\%$$

But what now is the **average growth rate** between 2020 and 2024?

Remember $k = 4$

$$\bar{\delta} = \left(\frac{250.42}{132.69} \right)^{1/4} - 1 \approx 0.1721 = 17.21\%$$

Interpretation: On average, between 2020 and 2023 y grew 17.21% every year.

Example

Note that

$$(1 + 17.21\%)^4 = (1 + 0.1721)^4 \approx 1.8873 \approx 1 + 88.73\%$$

? Growth Rates · Question 1

A variable grows 10% in one year and then falls 10% in the next. Over the two years together it:

- A. fell 1%
- B. rose 1%
- C. is unchanged
- D. fell 10%



A. Growth rates compound, they do not add:

$(1 + 0.1)(1 - 0.1) = 0.99$, so the variable ends 1% **below** where it started.

? Growth Rates · Question 2

A variable goes from 100 to 150 over 5 years. Its **average** growth rate is closest to:

A. 10%

B. 8.45%

C. 50%

D. 5%

☒ **B.** $\bar{\delta} = \left(\frac{150}{100}\right)^{1/5} - 1 = 1.5^{0.2} - 1 \approx 0.0845$, that is 8.45% per year.



Growth Rates · Question 3

A firm sold 240 units in 2020 and 300 units in 2023.

Compute:

- (i) the cumulative growth rate over the three years;
- (ii) the average annual growth rate.



Growth Rates · Solution

(i) $\delta_{2023|2020} = \frac{300-240}{240} = 0.25$, a cumulative 25%.

(ii) Here $k = 3$, so $\bar{\delta} = \left(\frac{300}{240}\right)^{1/3} - 1 = 1.25^{1/3} - 1 \approx 0.0772$, about **7.72% per year**.

Note 7.72% per year for 3 years is *not* 25/3, precisely because growth compounds.

Index Numbers: Introduction and Simple Index Numbers

Motivation



Which stock would you have purchased in 2016-10-03?

Motivation

What's the important question here?

Which one grew more! For this, levels are not as relevant as their evolution over time.

For example, say we have prices for two stocks A and B. Say we want to hold these stocks for only one period, which one would choose to invest your hard earned 💰?

How many stocks will you be able to buy of each stock, if their prices are a_t and b_t , and you have US\$1000?

$n_a = \frac{1000}{a_t}$ of stock A and $n_b = \frac{1000}{b_t}$ of stock B. Note that n_a and n_b are the amount of stock you can buy of each.

Motivation

Now is a new day ! Prices are now a_{t+1} and b_{t+1} , how much is your portfolio worth today?

1. If you bought stock A: $a_{t+1}n_a = a_{t+1} \frac{1000}{a_t}$
2. If you bought stock B: $b_{t+1}n_b = b_{t+1} \frac{1000}{b_t}$

Note that the 1000 is fixed, so it is just what we invested, it is not going to change anything in our decision on buying A vs B, what is really important is $\frac{a_{t+1}}{a_t}$ and $\frac{b_{t+1}}{b_t}$, that is the growth rate of prices for each stock, as

$$\frac{a_{t+1}}{a_t} = 1 + \delta_{t+1}^a \quad \text{and} \quad \frac{b_{t+1}}{b_t} = 1 + \delta_{t+1}^b$$

Motivation

You did not sell and...

Now is (another) new day ! Prices are now a_{t+2} and b_{t+2} , how much is your portfolio worth today?

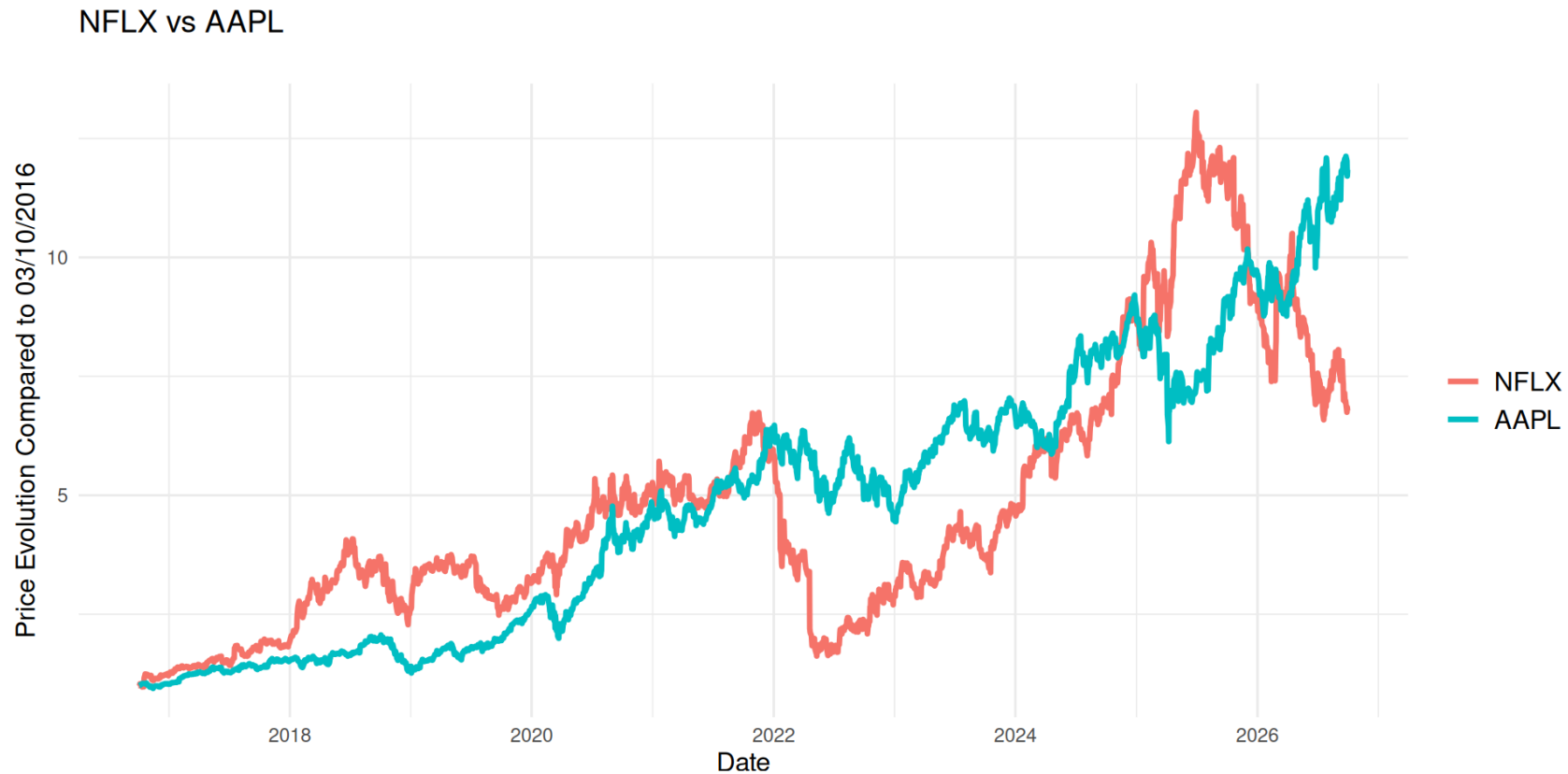
1. If you bought stock A: $a_{t+2}n_a = a_{t+2} \frac{1000}{a_t}$
2. If you bought stock B: $b_{t+2}n_b = b_{t+2} \frac{1000}{b_t}$

blah, blah...

$$\frac{a_{t+2}}{a_t} = 1 + \delta_{t+2}^a \quad \text{and} \quad \frac{b_{t+2}}{b_t} = 1 + \delta_{t+2}^b$$

If we wait two days, we only care how is the price today compared to when we made the purchase. We could make instead a plot of $1 + \delta_{t+k}$ to see which grew more!

Motivation



Now we have a much clear picture. We can compare their evolution starting in 2016-10-03.

What can you read from the plot?

Motivation

Note that for 2016-10-03, we would have, for A and B:

$$1 + \frac{a_t - a_t}{a_t} = 1 \quad \text{and} \quad 1 + \frac{b_t - b_t}{b_t} = 1$$

We just built an index number!

By convention though, instead of using 1 for the reference date, we will use 100 (as in 100%) for the reference day (we will put a name to this in a few moments).

When our new series was 2, we understood that $1 + \delta_{t+k} = 2$ or $\delta_{t+k} = 1$, *i.e.* the price grew 100%, or *doubled*! Now, with the new notation, we would have that the new series would take the value of 200, *i.e.* the price is now 200% the value it had at our initial date.

Index Numbers

Index Number

An **Index Number** is a scaled variable that takes as a reference point the value an underlying variable took at a fixed point in time. Say, for an underlying variable x_t we build an index I_t .

$$I_t = \frac{x_t}{x_0} \times 100$$

Where x_t is the value the underlying variable takes at time t , and x_0 represents the value this variable took at our reference point. This period is called **base**.

Note: In this case we used time as the dimension for t but this needs not be the case, we will see examples later.

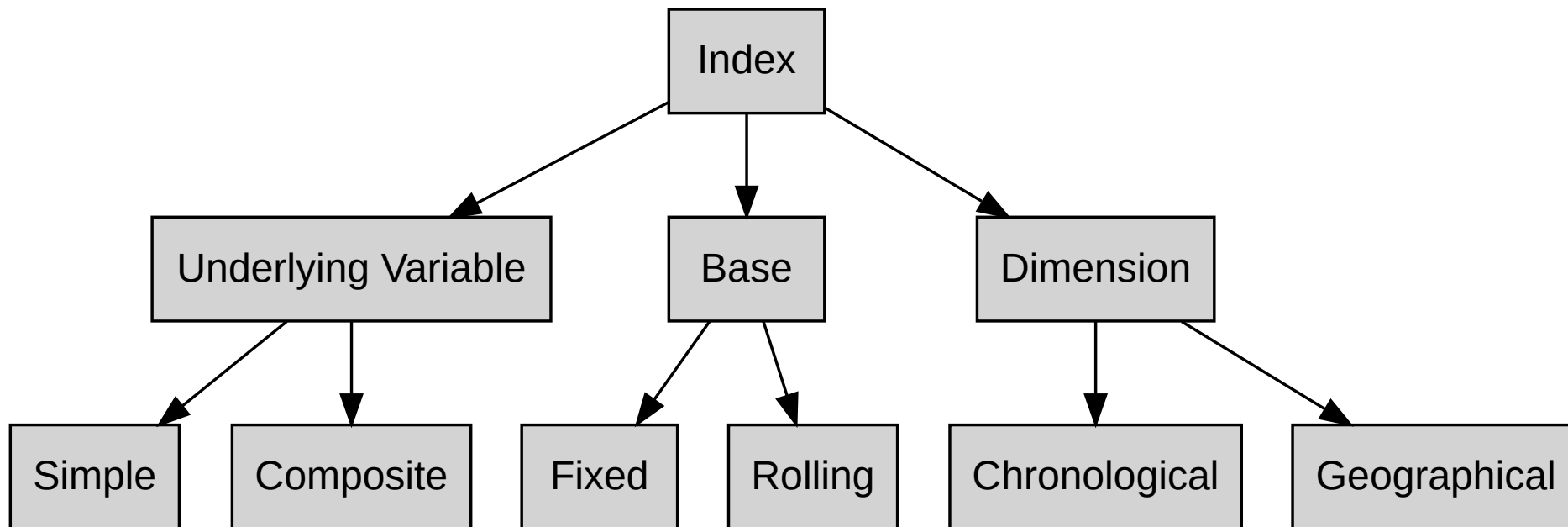
In our previous example, the base day would have been 2016-10-03. Note that in the base period

$$I_0 = \frac{x_0}{x_0} \times 100 = 100$$

Index Numbers

Index Numbers are defined by their underlying variable. This can change over time, or it can change according to another dimension, like geography, or both! We will see examples.

Also, an Index Number can have a fixed base or a moving base.



Index Numbers



Index Numbers

- Simple and Composite are mutually exclusive
- Fixed and Moving are mutually exclusive
- Rolling always needs a chronological dimension

Examples

Type	Example
Simple	Single stock price
Composite	SP500 (market index), CPI
Fixed Base	GDP, CPI, market index
Rolling Base	Some measures of GDP, Commodities Futures
Chronological	Stock price over time, CPI, GDP
Geographical	BigMac Index

Index Numbers

Importantly, an index number is *no bueno* to reflect the level of a variable, but its evolution or relative value against a base.