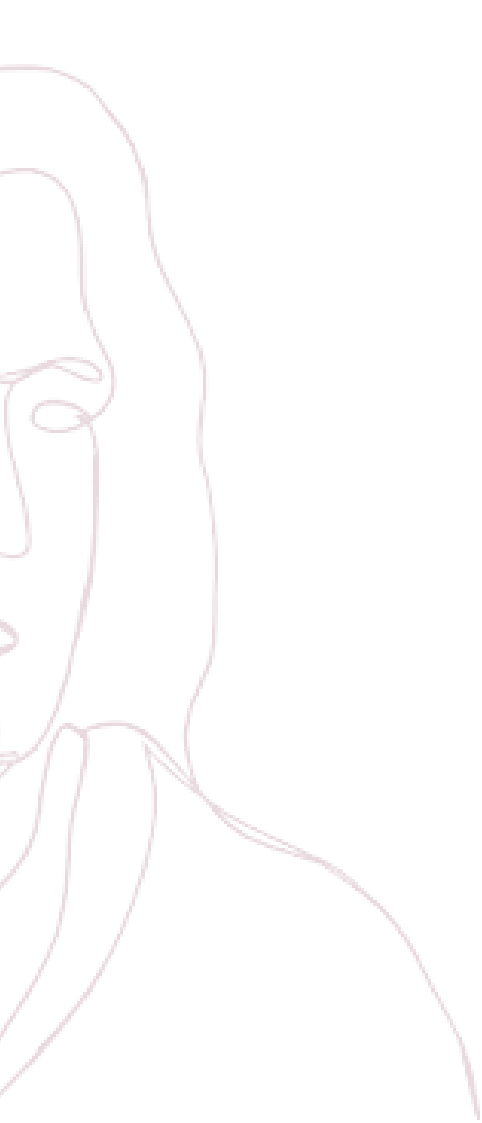




Simple and Composite Index Numbers

Statistics I · Lecture 2

Today

- Rolling base, chain and link indices
- Base change, and the tests for a good simple index
- Composite indices: baskets, Laspeyres, Paasche and Fisher

Index Numbers: Simple

Index Numbers (continued)

Rolling Base

Chain or Link Index

A **Chain or Link Index Number** is an index number that updates its base in every period.

$$I_t = \frac{x_t}{x_{t-1}} \times 100$$

Interpretation: It tells you how much the variable increased compared to the **previous** period.

Index Numbers

Index numbers allow us to easily compute the growth rate between any two periods, just like we would do with the underlying variable.

1. Simple Index Number

$$1 + \delta_{t+k,t} = \frac{x_{t+k}}{x_t} = \frac{\frac{x_{t+k}}{x_0}}{\frac{x_t}{x_0}} = \frac{I_{t+k}}{I_t}$$

Index Numbers

2. Chain Index Number

$$1 + \delta_{t+k,t} = \frac{x_{t+k}}{x_t} = \frac{x_{t+k}}{x_{t+k-1}} \frac{x_{t+k-1}}{x_{t+k-2}} \dots \frac{x_{t+1}}{x_t}$$

$$1 + \delta_{t+k,t} = I_{t+k} \times I_{t+k-1} \times \dots \times I_{t+1}$$

Index Numbers



Do not use 100 index when multiplying chain or link indices, use the decimal notation.

Note that if the underlying variable does not change at all, you should expect the same value, $x_t = x_{t+1} = x_{t+2}$

In this case $I_{t+1} = \frac{x_{t+1}}{x_t} = 1$ and $I_{t+2} = \frac{x_{t+2}}{x_{t+1}} = 1$, and therefore

$$1 + \delta_{t+2,t} = 1 \times 1 = 1$$

This is the correct way to do it .

Index Numbers

⚠ And if we multiplied the indices with the 100?

If instead you used the 100 to multiply:

$$1 + \delta_{t+2,t} = 100 \times 100 = 10000$$

and we get that x grew 100 times! (✗ because we know that $x_t = x_{t+1} = x_{t+2}$)

Index Numbers

We can always transform a chain index number into a fixed base index number.

1. Chose a reference, for example for period 3.
2. Remember: what you want to achieve: $\forall t, I'_t = \frac{x_t}{x_3}$
3. Remember: that we have $I_t = \frac{x_t}{x_{t-1}}$
4. Use what we did just before:

$$I'_t = I_t I_{t-1} \dots I_4 = \frac{x_t}{x_{t-1}} \frac{x_{t-1}}{x_{t-2}} \dots \frac{x_4}{x_3} = \frac{x_t}{x_3} = I'_t$$

Example

Suppose we want to convert this Chain Index into Fix Base index, with the base year in 2021. This will be our 100.

	2021	2022	2023	2024	2025	2026
Price	30	34	37	35	36	39
		$\frac{34}{30} 100$	$\frac{37}{34} 100$	$\frac{35}{37} 100$	$\frac{36}{35} 100$	$\frac{39}{36} 100$
I_{chain}		113.3	108.8	94.6	97.1	108.3
		$100 \frac{113.3}{100}$	$113.3 \frac{108.8}{100}$	$123.3 \frac{94.6}{100}$	$116.7 \frac{97.1}{100}$	$120 \frac{108.3}{100}$
$I_{t 2021}$	100	113.3	123.3	116.7	120	130

Example

Note that, for example in 2025, we have $I_{2025|2021} = 116.7$, which was computed as $123.3 \times \frac{94.6}{100}$. But 123.3 was the base 2021 index of the previous period, which in turn was computed as $113.3 \frac{108.8}{100}$, so in reality:

$$116.7 = 123.3 \frac{94.6}{100} = 113.3 \frac{108.8}{100} \frac{94.6}{100} = 100 \frac{113.3}{100} \frac{108.8}{100} \frac{94.6}{100}$$

If you pay attention, you will see that we have there the product of the *chain Index* in 2022, 2023, and 2024. We divided by 100 because, to multiply, we need to divide by 100 before.

Base Change

In case we have the data, we just need to compute the Index with the base that we want, however, if we do not have the data, and we just have a Fix Base index, we can do it all the same!

Say we have an index with base 0. Maybe we want to change it to base 2.

Base Change

Table 1: Changing Base

Base 0

period	Index	Formula
0	$I_{0 0}$	$\frac{x_0}{x_0}$
1	$I_{1 0}$	$\frac{x_1}{x_0}$
2	$I_{2 0}$	$\frac{x_2}{x_0}$
3	$I_{3 0}$	$\frac{x_3}{x_0}$
...	...	...
t	$I_{t 0}$	$\frac{x_t}{x_0}$

Base 2

period	Index	Formula
0	$I_{0 2}$	$\frac{x_0}{x_2}$
1	$I_{1 2}$	$\frac{x_1}{x_2}$
2	$I_{2 2}$	$\frac{x_2}{x_2}$
3	$I_{3 2}$	$\frac{x_3}{x_2}$
...	...	...
t	$I_{t 2}$	$\frac{x_t}{x_2}$

Base Change

But how can we transform them, if we do not know $x_i \forall i$?

$$I_{t|2} = \frac{x_t}{x_2} = 1 \frac{x_t}{x_2} = \frac{1/x_0}{1/x_0} \frac{x_t}{x_2} = \frac{\frac{x_t}{x_0}}{\frac{x_2}{x_0}} = \frac{I_{t|0}}{I_{2|0}}$$

And that is how we can re-base an index without knowing the value for the underlying variable (x_t)!



Mixing up stuff

Previously we covered how to go from a chain index to a fixed base one. Now we saw how to change the base of a fixed base index. With both things together you can turn a chain index into fixed base one, with base you want!

Example

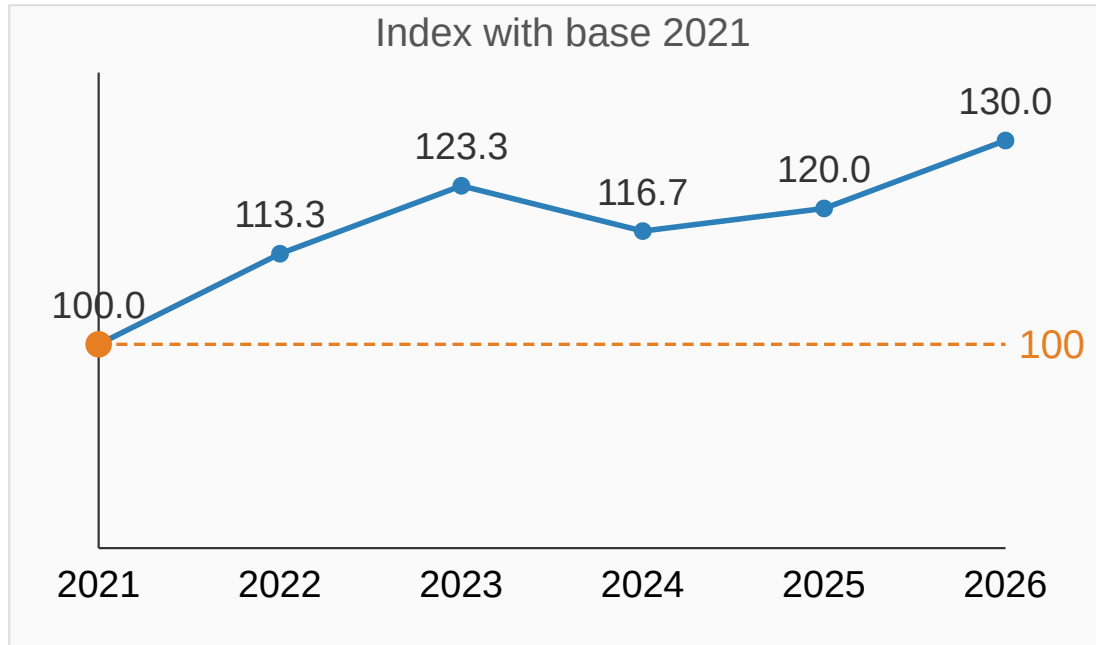
Stock prices

	2021	2022	2023	2024	2025	2026
Price	30	34	37	35	36	39
$I_{t 2021}$	100	113.3	123.3	116.7	120	130
	$\frac{100}{123.3}$	$\frac{113.3}{123.3}$	$\frac{123.3}{123.3}$	$\frac{116.7}{123.3}$	$\frac{120.0}{123.3}$	$\frac{130.0}{123.3}$
$I_{t 2023}$	81.1	91.9	100	94.6	97.3	105.4

New information? Yes! For example, without any computation we know that the price in 2026 was 30% larger than in 2021, and 5.4% larger than in 2023

Does changing the base change the story?

Same prices as above. Move the slider to pick which year is the base, and watch what moves and what does not.



Base year: **2021**



Growth rates

2022: 13.3%

2023: 8.8%

2024: -5.4%

2025: 2.9%

2026: 8.3%

These never change.

What the chart showed

Changing the base **slides the whole curve up or down**: the orange 100 line stays where it is, and a different year gets pinned to it.

But the **shape is untouched**, and every growth rate in the panel stays exactly the same. Re-basing changes the *yardstick*, never the underlying evolution.

If two countries publish the same price series on different bases, they are not disagreeing about inflation. Divide one index by another of the same series and the base cancels, which is precisely the formula we just derived.

Tests for good Simple Index Numbers

We can test the properties for Simple Index Numbers. These represent desirable properties for good index numbers.

1. **Identity test:** If the variable in the current period is the same as in the base period, the index should be 1 (or 100). Of course this implies

$$I_{0|0} = 1$$

.

2. **Proportionality test:** If the variable increases by k , the index should increase by k .

$$x_t = kx_0 \Rightarrow i_{t|0} = k$$

Tests for good Simple Index Numbers

3. **Time Reversal test:** If you exchange the base between two periods, then the index in one base, should be the inverse of the other:

$$i_{t|0} = \frac{1}{i_{0|t}} \Leftrightarrow i_{t|0} i_{0|t} = 1$$

4. **Factor Reversal test:** If we have two index numbers, one for variable x and another for y , then if $z = x \times y$ we should have

$$i_{t|0}^z = i_{t|0}^x \times i_{t|0}^y$$

Tests for good Simple Index Numbers

5. **Circularity test:** The index at t base 0 should be the product of all the link index numbers up to t ,

$$i_{t|0} = i_{t|t-1} \times i_{t-1|t-2} \cdots i_{2|1} \times i_{1|0}$$

This formalizes the relationship between fixed base and chain index numbers.

6. **Homogeneity test:** If the variable, for every t is multiplied by a constant, the index should not be affected.

Tests for good Simple Index Numbers

7. **Well-definedness:** The index number must be meaningful, and as prices and quantities usually are larger than zero, index numbers must be positive. Even further, a good index number should not take the value of ∞ .

Ask yourself, what information are you getting here?

A value of 0 might also be problematic, although it is not totally meaningless. The problem arises when you want to deal with chain index numbers, where a 0 would indeterminate the rest of the chain. Also, you could never use as a base period, a period where the variable took the value of 0.

? Simple Index Numbers · Question 1

A simple price index with base 2020 reads 125 in 2024. This tells you that the price:

A. rose 125%

B. rose 25%

C. is 125 euros

D. nothing, without the base price

✓ **B.** The index is $\frac{x_t}{x_0} \times 100$. A reading of 125 means $\frac{x_t}{x_0} = 1.25$, so the price is 25% **above** the base.

? Simple Index Numbers · Question 2

To turn a chain (link) index into a fixed base index you:

- A. multiply them, in decimal notation
- B. add the link indices
- C. average them
- D. subtract 100 from each

 **A.** Link indices multiply, and you must divide by 100 first, otherwise each extra factor of 100 inflates the answer.



Simple Index Numbers · Question 3

The price of a good was 30 in year 1, 34 in year 2 and 37 in year 3.

Compute the fixed base index with base year 1 for all three years, and the link index for year 3.



Simple Index Numbers · Solution

Fixed base year 1: $I_{1|1} = 100$, $I_{2|1} = \frac{34}{30} \times 100 = 113.3$, $I_{3|1} = \frac{37}{30} \times 100 = 123.3$.

Link index for year 3: $i_{3|2} = \frac{37}{34} \times 100 = 108.8$.

Check: $100 \times \frac{113.3}{100} \times \frac{108.8}{100} \approx 123.3$  (chain the link indices in decimal, never on the 100 scale)

Index Numbers: Composite Index Numbers

Composite Index Numbers

Composite Index Numbers (a.k.a. Aggregate Index Numbers)

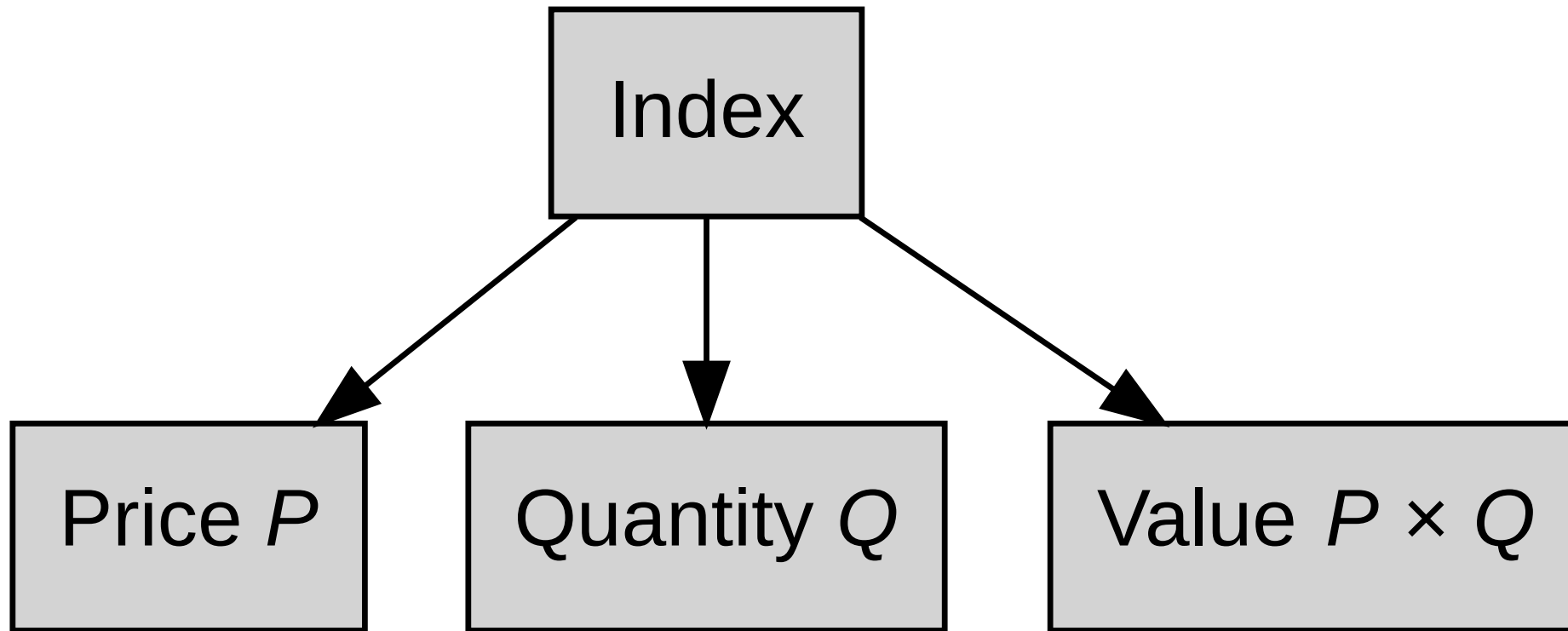
Composite Index Numbers are index numbers that have more than a single underlying variable, or which underlying variable is computed itself as a combination of other variables.

A composite index tracks the combined evolution of multiple phenomena at the same time. It summarizes how a diverse group of elements changes over time.

Example: The performance of a class, measured by the average grade. The average is computed with each individual grade, and therefore if we make an index based on the average of the class, we would have a composite index number.

Composite Index Numbers

This is a great opportunity to introduce another classification for our index numbers:



Composite Index Numbers

Typically, values are collected in nominal terms, *i.e.* **current prices**, which is a *value*.

$$v_t = p_t \times q_t$$

Note that, if δ_{t+1}^v is the growth rate of v_t :

$$v_{t+1} = v_t (1 + \delta_{t+1}^v)$$

and obviously, it is also true that:

$$v_{t+1} = p_{t+1} \times q_{t+1}$$

Composite Index Numbers

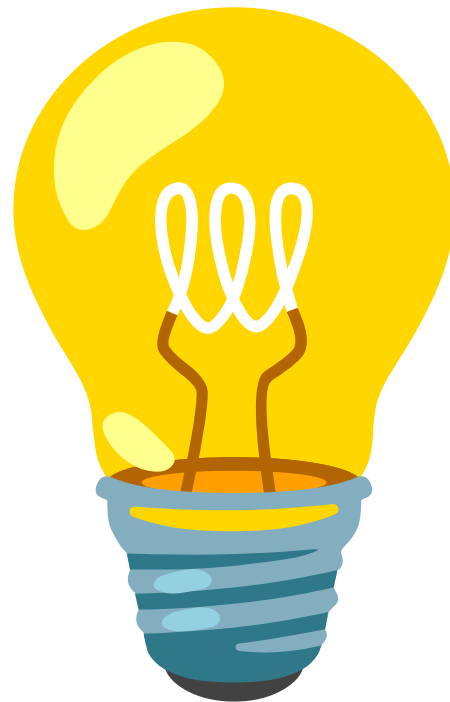
But if we let δ_{t+1}^p be the growth rate of prices, and δ_{t+1}^q the growth rate of quantities we can rewrite v_{t+1} as:

$$v_{t+1} = p_t (1 + \delta_{t+1}^p) q_t (1 + \delta_{t+1}^q)$$

$$v_{t+1} = p_t q_t (1 + \delta_{t+1}^p)(1 + \delta_{t+1}^q)$$

$$v_{t+1} = v_t (1 + \delta_{t+1}^p)(1 + \delta_{t+1}^q)$$

$$v_t (1 + \delta_{t+1}^v) = v_t (1 + \delta_{t+1}^p)(1 + \delta_{t+1}^q)$$



Composite Index Numbers

$$(1 + \delta_{t+1}^v) = (1 + \delta_{t+1}^p)(1 + \delta_{t+1}^q)$$

δ_t^v is what we call **nominal change rate**, while δ_t^q is what we call **real change rate**. We can find out the real change rate (δ^q), if we know the nominal change rate (δ^v), and the prices change rate (δ^p):

$$(1 + \delta_t^q) = \frac{(1 + \delta_t^v)}{(1 + \delta_t^p)}$$

$$\delta_t^q = \frac{(1 + \delta_t^v)}{(1 + \delta_t^p)} - 1$$

Composite Index Numbers

Suppose we have index numbers for v , q , and p , all with base b (remember you can re-base your index numbers easily if they do not share the same base).

$$I_{t|b}^q = \frac{I_{t|b}^v}{I_{t|b}^p}$$

$$\frac{q_t}{q_b} = \frac{\frac{v_t}{v_b}}{\frac{p_t}{p_b}}$$

$$\frac{q_t}{q_b} = \frac{\frac{p_t q_t}{p_b q_b}}{\frac{p_t}{p_b}} = \frac{p_t q_t}{p_b q_b} \frac{p_b}{p_t} = \frac{q_t}{q_b}$$

Example

	2022	2023	2024	2025	2026
Sales Index	99	100	107	110	111
Price Index	95	100	102	106	109

Let's find the quantities index, with base 2023. Immediately we can fix $i_{t|b}^q = 100$.

$$I_{2022}^q = \frac{I_{2022}^v}{I_{2022}^p} = \frac{99}{95} = 1.04$$

Multiplying by 100, $I_{2022}^q = 104$, so quantities decreased from 2022 to 2023!

Example

	2022	2023	2024	2025	2026
Quantities Index	104.21	100	104.9	103.77	101.83

Now we can compute what was the real change rate between 2025 and 2026:

$$\delta_{2026} = \frac{102}{104} - 1 = -0.0187 = -1.87\%$$

Real sales decreased 1.87% from 2025 to 2026.

Example

Let's check:

$$\delta_{2026}^q = \frac{(1 + \delta_{2026}^v)}{(1 + \delta_{2026}^p)} - 1 = \frac{111/110}{109/106} - 1 = 0.9813 - 1$$

$$\delta_{2026}^q = -0.0187 = -1.87\%$$

Baskets

Basket 🛒

Basket are sets of goods, which are used to track the evolution of, for example, a price level.

In economics, we often use price and quantity index numbers to measure how the prices or quantities of a set of goods (a **basket**) change over time.

A very important question: What are the weights that the different goods should have in the basket?

Aggregate Index Numbers

From the family of formulas to create weighted composite index numbers, we will focus on the two most common:

1. **Laspeyres**: uses base-period quantities (prices) as weights.
2. **Paasche**: uses current-period quantities (prices) as weights.

These are used to measure the evolution of *prices* (*quantities*).

Kids' Explanation

Old basket ( *past*):

-  10 apples
-  5 oranges

Old prices ( *past*):

- $p_{\text{apple}} = 1$
- $p_{\text{orange}} = 0.8$

New basket ( *today*):

-  8 apples
-  6 oranges

New prices ( *today*):

- $p_{\text{apple}} = 1.2$
- $p_{\text{orange}} = 1.1$

Let's consider that the **base year is past**.

Kids' Explanation - Prices

Laspeyres - Price:

Use always 🛒_{past}, but relevant 🏷️! Laspeyres Price Index (LPI):

$$\text{🛒}_{past} \mid \text{🏷️}_{past} \quad LPI_{past} = \frac{10 \times 1 + 5 \times 0.8}{10 \times 1 + 5 \times 0.8} = \frac{14}{14} = 1$$

$$\text{🛒}_{past} \mid \text{🏷️}_{today} \quad LPI_{today} = \frac{10 \times 1.2 + 5 \times 1.1}{10 \times 1 + 5 \times 0.8} = \frac{17.5}{14} = 1.25$$

Remember that 1 corresponds to 100, and 1.25 corresponds to 125.

Kids' Explanation - Prices

Paasche - Price:

Use always 🛒_{today}! Paasche Price Index (PPI):

$$\text{🛒}_{today} | \text{🏷️}_{past} \quad PPI_{past} = \frac{8 \times 1 + 6 \times 0.8}{8 \times 1 + 6 \times 0.8} = \frac{12.8}{12.8} = 1$$

$$\text{🛒}_{today} | \text{🏷️}_{today} \quad PPI_{today} = \frac{8 \times 1.2 + 6 \times 1.1}{8 \times 1 + 6 \times 0.8} = \frac{16.2}{12.8} = 1.27$$

Remember that 1 corresponds to 100, and 1.27 corresponds to 127.

In general - Laspeyres

1. For prices, how much the old basket would cost at current prices relative to the **old basket and prices**?

$$LPI = \frac{\sum_{k=1}^m p_t^k q_0^k}{\sum_{k=1}^m p_0^k q_0^k}$$

2. For quantities, how much the current basket would cost at the old prices, relative to the **old basket and prices**?

$$LQI = \frac{\sum_{k=1}^m p_0^k q_t^k}{\sum_{k=1}^m p_0^k q_0^k}$$

Note: denominator is the same.

In general - Paasche

1. For prices, how much more expensive is the **current basket** at current prices, relative to the original prices.

$$PPI = \frac{\sum_{k=1}^m p_t^k q_t^k}{\sum_{k=1}^m p_0^k q_t^k}$$

2. For quantities, how much more the **current basket** costs, relative to what would cost the original basket at current prices.

$$PQI = \frac{\sum_{k=1}^m p_t^k q_t^k}{\sum_{k=1}^m p_t^k q_0^k}$$

Note: numerator is the same.

Conclusions

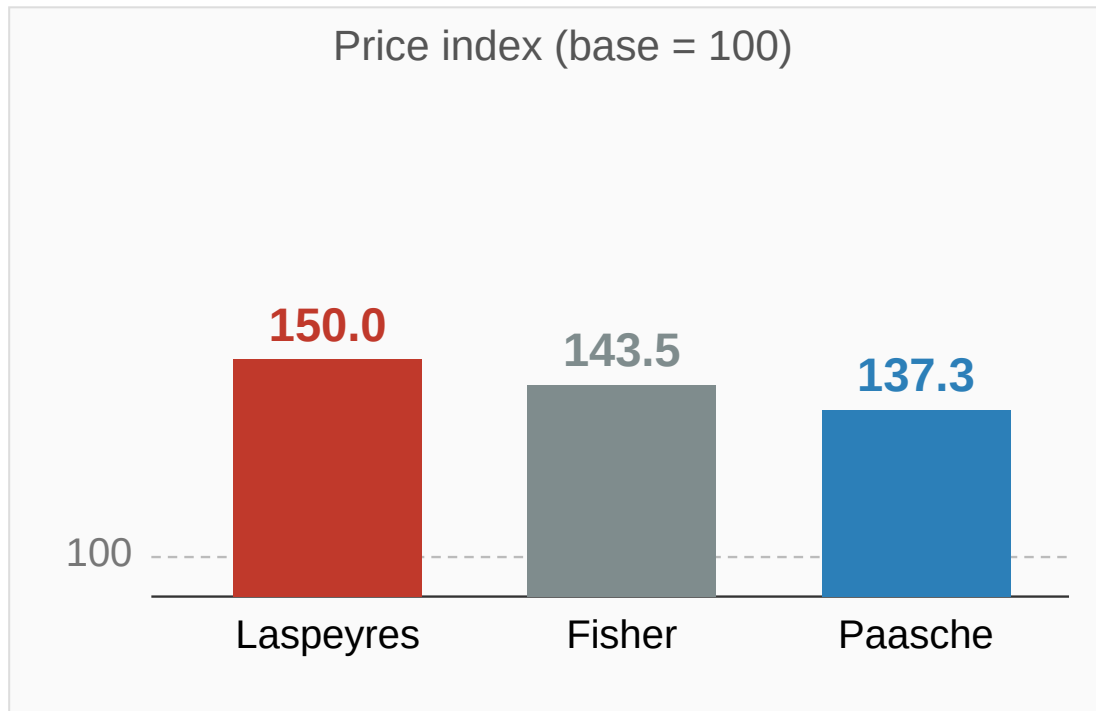
1. In both index numbers, for prices, quantities are constant: All variation comes from prices.
2. In both index numbers, for quantities, prices are constant: All variation comes from quantities.

We tend to use much more often *Laspeyres* because quantities take much longer to be accurately collected than prices. Ergo, national statistics institutes usually use Laspeyres price index to measure the price evolution.

Do the two indices agree?

Two goods, both costing 1 at the base period, both bought 10 times. Now good 1 gets more expensive, and people react by buying less of it.

Left slider: how much more expensive good 1 becomes. **Right slider:** how strongly people substitute away from it.



Price of good 1: **2.00×**



Substitution: **0.50**



Basket

good 1: 10 → 7.1

good 2: 10 → 11.9

Gap $L - P = 12.7$

What the chart showed

With **no substitution** the two indices are identical: the basket never changed, so it does not matter whose basket you weight with.

As substitution grows, **Laspeyres stays put** (it always weights with the *old* basket, which still contains a lot of the good that got dearer) while **Paasche falls** (it weights with the *new* basket, which has already moved away from that good).

So Laspeyres tends to **overstate** the rise in the cost of living, and Paasche tends to **understate** it. The truth is somewhere in between, which is exactly the gap the next index tries to close.

Fisher Index

Some authors argue, however, that the true measure for *cost of life* is in between both metrics. This is captured by the **Fisher Index**:

$$FPI_{t|0} = \sqrt{LPI_{t|0} \times PPI_{t|0}}$$

$$FQI_{t|0} = \sqrt{LQI_{t|0} \times PQI_{t|0}}$$

Example: the data

A basket of 15 goods, observed in 2024, 2025 and 2026. Each cell is what the basket costs with the **row** year's prices and the **column** year's quantities, $\sum_{k=1}^{15} p_{row}^k q_{col}^k$.

	q_{2024}	q_{2025}	q_{2026}
p_{2024}	8562	9210	9920
p_{2025}	9150	9584	
p_{2026}	9657		10013

$$PPI_{2026|2024} = 100 \times \frac{10013}{9920}$$

Example: LPI, PPI and FPI

What are the three indices if we choose 2024 as the base year?

t	2024	2025	2026
$LPI_{t 2024}$	100	106.9	112.8
$PPI_{t 2024}$	100	104.1	100.9
$FPI_{t 2024}$	100	105.5	106.7

$$FPI_{2025|2024} = \sqrt{106.9 \times 104.1}$$

? Composite Index Numbers · Question 1

The Laspeyres price index weights prices using quantities from:

- A. the current period
- B. the base period
- C. the average of both
- D. neither, it uses no quantities

✓ **B.** Laspeyres fixes the **base period** basket, $LPI_{t|0} = \frac{\sum_k p_t^k q_0^k}{\sum_k p_0^k q_0^k}$.

Paasche is the one that uses current quantities.

? Composite Index Numbers · Question 2

The Fisher index is:

- A. the arithmetic mean of Laspeyres and Paasche
- B. the geometric mean of Laspeyres and Paasche
- C. always between 100 and 200
- D. always equal to Laspeyres

✓ **B.** $FPI_{t|0} = \sqrt{LPI_{t|0} \times PPI_{t|0}}.$



Composite Index Numbers · Question 3

For a basket you computed $LPI_{t|0} = 110$ and $PPI_{t|0} = 104$.

Compute Fisher's price index, and explain why it must always fall between the other two.



Composite Index Numbers · Solution

$$FPI_{t|0} = \sqrt{110 \times 104} = \sqrt{11440} \approx 106.96$$

Fisher is the **geometric mean** of the other two, and the geometric mean of two positive numbers always lies between them. So whenever $LPI \neq PPI$, Fisher sits strictly in between, here between 104 and 110.