



Sets, Events and the Concept of Probability

Statistics I · Lecture 4

Today

- The concept of probability
- A refresh on set theory
- Sample space and events
- Laplace and frequency interpretations

Disclaimer

These slides are a free translation and adaptation from the slide deck for Estatística I by Prof. Sandra Custódio and Prof. Teresa Ferreira from the Lisbon Accounting and Business School, Polytechnic University of Lisbon.

Probability, background

The concept

Consider the following scenario:

 **Investor:** What's the probability this startup will succeed?

 **Analyst:** Hard to say, every startup is different.

 **Investor:** But if you had to guess, based on similar cases?

 **Analyst:** Maybe 1 in 3 succeed under these conditions.

 **Investor:** So, would you bet on it?

 **Analyst:** Yes, I would.

 **Investor:** Even if the odds aren't great?

 **Analyst:** I believe this one has what it takes.

The concept

Here we can define probability in terms of **frequency of occurrence**, *i.e.* as a percentage of successes in a moderately large number of similar situations.

This is the most natural and traditional way of thinking about probability.

- Regarding a fair coin  we could say: “with probability 50% the coin lands on heads” meaning “roughly half of the time.”

But, what if this company belongs to a completely novel market sector?

The concept

There might be situations where the frequency concept is not adequate, because it might refer to a one-time event. These are **subjective beliefs**.

- A company is recruiting a new CEO, and a board member says:
“I believe there’s a 90% chance that our chosen candidate will be an effective CEO.”

The concept

It might seem easy to disregard the second case as unscientific or useless. However, many times people need to make decisions under uncertainty with not enough data (or no data at all!) about previous realizations of the specific event.

Beliefs allow the decision maker to, well, make *some* decision, at least consistently.

- What's the difference between both situations?
- What do they have in common?

Uncertainty

A refresh on Set Theory

Sets and elements

A **set** is a collection of objects, which are **elements** of the set.

Definition

Let S represent a set, and s an element of that set, we write $s \in S$ to mean s **belongs** to S .

If s does not belong to S , we write $s \notin S$.

Definition

If a set S does not have any element, then it is the **empty** set, denoted by $\emptyset$.

How to write down a Set

There are several ways to specify a set.

By extension, or as a list:

- If a set S has a finite number of elements ($x_i \in S$) we can write it like this:

$$S = \{x_1, x_2, \dots, x_n\}$$

- If a set S has an infinite (but countable) elements ($x_i \in S$) we can write it like:

$$S = \{x_1, x_2, \dots\}$$

How to write down a Set

By describing the property (P) that x must satisfy to be included in S :

$$S = \{x | x \text{ satisfies } P\}$$

in this case $|$ reads as *such that*. For example

$$S = \{x \in \mathbb{R} \mid x \geq 0\}$$

to describe the non-negative real numbers.

This example is special, as the positive real numbers cannot be written down as a list. In this case the interval $[0, \infty)$ is an **uncountable** set.

More definitions

Definition

If $\forall x \in S$ it is also true that $x \in T$, then we say that S is a **subset** of T , and we write it like $S \subseteq T$.

Definition

If $S \subseteq T$ and at the same time $T \subseteq S$ then we say that S and T are **equal**, and we write it $S = T$.

Definition

The **universal set** Ω is the set that contains all objects that could conceivably be of interest in a particular context.

By definition, any set S must be a subset of Ω .

More definitions

The universal set is important because it defines the scope of our analysis. Say we are studying the performance of students of Statistics I in 2026.

The cars parked outside our institution do not belong to the universal set, because they are not relevant for our purpose. Only students of Statistics I in 2026 belong to the universal set.

Set Operations

Definition

The **complement** of a set S , with respect to Ω , is the set $\{x \in \Omega | x \notin S\}$, that is, all the relevant elements that do not belong in S . We denote it as S^c .

Corollary: It is easy to see that $\Omega^c = \emptyset$.

Set Operations

Definition

The **union** of two sets S and T is the set of all elements that belong to S *or* T (or both), and is denoted by $S \cup T$.

$$S \cup T = \{x \in \Omega | x \in S \vee x \in T\}$$

Definition

The **intersection** of two sets S and T is the set of all elements that belong to S *and* T , and is denoted by $S \cap T$.

$$S \cap T = \{x \in \Omega | x \in S \wedge x \in T\}$$

Note that $\vee$ stands for *or*, and $\wedge$ stands for *and*.

Set Operations

Sometimes we might need to consider the union or intersection of many sets, and for that we can use a notation similar to the one we used for summations:

- $\bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup \dots = \{x \in \Omega \mid x \in S_n \text{ for some } n\}$
- $\bigcap_{n=1}^{\infty} S_n = S_1 \cap S_2 \cap \dots = \{x \in \Omega \mid x \in S_n \text{ for every } n\}$

Set Operations

Definition

Two sets (say S and T) are said to be **disjoint** if $S \cap T = \emptyset$.

More generally, a collection of sets S_n is disjoint if S_i and S_j are disjoint when $i \neq j$.

Definition

A collection of sets is said to be a **partition** of a set S if the sets in the collection are:

- Disjoint
- Their union is S

The set of all subsets of Ω is called the **power set**, denoted $\mathcal{P}(\Omega)$. A partition of Ω is then a collection $\{A_i\} \subseteq \mathcal{P}(\Omega)$.

Set Definition

The number of elements of a set S is known as its **cardinality** and it is denoted as $\#S$. $\#S$ satisfies:

1. $\#S \geq 0$
2. $\#\emptyset = 0$

Set definition

If we have two sets, S and T , we define the operation *set minus* as $\setminus$ the set that contains all elements of S that do not belong to T

$$S \setminus T = \{x \in S | x \notin T\}$$

? Set Theory · Question 1

Two sets S and T are **disjoint** when:

A. $S \cup T = \emptyset$

B. $S = T$

C. $S \cap T = \emptyset$

D. $S \subseteq T$

 **C.** Disjoint means they share no element at all, so their **intersection** is empty. Their union is empty only if both sets are.

? Set Theory · Question 2

The set $S \setminus T$ contains exactly the elements that:

A. belong to both S and T

B. belong to T but not to S

C. belong to neither

D. belong to S but not to T

✓ **D.** $S \setminus T = \{x \in S \mid x \notin T\}$. Note the operation is not symmetric: $S \setminus T$ and $T \setminus S$ are different sets.



Set Theory · Question 3

Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A = \{1\}$, $B = \{3, 6\}$ and $C = \{2, 4, 6\}$.

Write down $A \cup B$, $B \cap C$, $(B \cup C)^c$ and $C \setminus B$.



Set Theory · Solution

$$A \cup B = \{1, 3, 6\}$$

$$B \cap C = \{6\}$$

$$B \cup C = \{2, 3, 4, 6\}, \text{ so } (B \cup C)^c = \{1, 5\}$$

$$C \setminus B = \{2, 4\}$$

Back to Probability

In probability, Ω , the universal set, is a non-empty set that contains all possible **outcomes** of an experiment. Each outcome is represented by ω , and obviously $\omega \in \Omega$.

Back to Probability

The sample space (Ω) can be:

- Discrete, when $\#\Omega$ is finite, or countable infinite.
- Continuous, when $\#\Omega$ is uncountable.

Back to Probability

Consider the experiment of throwing a die  and noting the number shown on side facing upwards.

1. The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$
2. In this case $\#\Omega = 6$
3. Ω is discrete.

Back to Probability

Consider now the random experiment of measuring the life expectancy of a lamp  , measured in hours.

1. The sample space is $\Omega = \{x \in \mathbb{R} | x \geq 0\}$
2. In this case, Ω is all non-negative real numbers.
3. Ω is continuous.

Remember, Ω must include all possible outcomes from your experiment!
Even then ones that seem ludicrous.

Back to Probability

Definition

A subset A of the sample space Ω is called an **event**.

$$A \subseteq \Omega$$

By definition then Ω is also an event.

Definition

We call the **realization** of an event A if, after an experiment, outcome ω is realized, and $\omega \in A$.

Example

Let's go back to our experiment with the 

The sample space is: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Within this space, we can define the following events:

1. $A = \{1, 3, 5\}$, *i.e.* the number is odd.
2. $B = \{3, 4, 5, 6\}$, *i.e.* the number is at least 3.
3. $C = \{1, 2, 3\}$, *i.e.* the number is lower than 4.
4. $D = \{6\}$, *i.e.* the number is larger than 5.

Example

Now let's revisit the example of our 

The sample space is: $\Omega = \{x \in \mathbb{R} | x \geq 0\}$

In this space, we can define the following events:

1. $A = \{x \in \mathbb{R} | 75 < x < 95\}$, i.e. the  lasts between 75 and 95 hours.
2. $B = \{x \in \mathbb{R} | x \leq 100\}$, i.e. the  lasts no longer than 100 hours.
3. $C = \{x \in \mathbb{R} | x \geq 60\}$, i.e. the  lasts at least 60 hours.

Events

Definitions

- An **elementary event** is any event that contains a single element (*i.e.* $\#A = 1$)
- An **impossible event** is an event with no outcome, (*i.e.* $\#A = 0$). As a consequence, an impossible event coincides with the empty set $\emptyset$.
- A **certain event** is indeed the event Ω , as for any outcome we obtain ω , this outcome belongs to the sample space Ω by definition.

Mixing up Sets and Probability

Consider two events A and B both subsets of Ω

1. A^c contains all the outcomes that are not in A . A^c is the event of *not* A .
2. If $A \subseteq B$, then an outcome that realizes event A ($\omega \in A$), also realizes B , as $A \subseteq B \Rightarrow \omega \in B$ as well. $A \Rightarrow B$
3. For $A \cup B$ to happen, we need $\omega \in A$ **or** $\omega \in B$, which means that A happens, or B happens, or both happen simultaneously.

Mixing up Sets and Probability

4. For $A \cap B$ to happen, we need $\omega \in A$ **and** $\omega \in B$, which means that A and B happen simultaneously.
5. A and B are incompatible if $A \cap B = \emptyset$, *i.e.* if an outcome is in one set, it cannot be in another, for example it cannot be that A and A^c happen simultaneously!

Inherited set properties for events

Consider two events A and B both subsets of Ω

1. Commutativity: $A \cup B = B \cup A$; $A \cap B = B \cap A$
2. Associativity: $(A \cup B) \cup C = A \cup (B \cup C)$;
 $(A \cap B) \cap C = A \cap (B \cap C)$
3. Distributivity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. De Morgan's Laws: $(A \cap B)^c = A^c \cup B^c$; $(A \cup B)^c = A^c \cap B^c$

Inherited set properties for events

5. $(A^c)^c = A$

6. Complement law: $A \cup A^c = \Omega$; $A \cap A^c = \emptyset$

7. Identity element: $A \cup \emptyset = A$; $A \cap \Omega = A$

8. Absorbing element: $A \cup \Omega = \Omega$; $A \cap \emptyset = \emptyset$

9. Idempotent law: $A \cup A = A$; $A \cap A = A$

10. $A \subset B \Rightarrow A \cap B = A$; $A \subset B \Rightarrow A \cup B = B$

Example

Consider the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$, from our  case.

Define the events:

$$A = \{1\}, \quad B = \{3, 6\}, \quad C = \{2, 4, 6\}, \quad D = \{4, 5, 6\}$$

Example

Let's define the following events in Ω

- $A \cup B$
- $A \cap B$
- A^c
- $(A \cup B)^c$
- $(B \cap C)^c$
- $B \setminus C$
- $C \setminus D$

Concept of probability

Besides the concepts we already saw of frequency and subjectivity for probability, there was an older, called “*classic*” one. This one was introduced by Pierre-Simon **Laplace** in 1812.

Laplace or Classic interpretation of probability

Let A be an event defined over a finite Ω . The probability of event A is defined as:

$$P(A) = \frac{\#A}{\#\Omega}$$

Example

Consider now an experiment throwing two dice 🎲 🎲

1. How many outcomes are in Ω ? $6^2 = 36$, $\#\Omega = 36$.
2. Let A be the event where both dice show the same number:

$$A = \{(1, 1), (2, 2), \dots, (6, 6)\}$$

here $\#A = 6$

3. The probability that both dice show the same number is

$$P(A) = \frac{\#A}{\#\Omega} = \frac{6}{36} = \frac{1}{6}$$

Laplace or Classic interpretation of probability

The problem with this interpretation, is that we cannot use it, or it becomes meaningless, if when Ω is uncountable or infinite. Also, what if the outcomes are not equally likely? (*i.e.* if the dice are not fair?)

Frequency interpretation

This is today still the dominant interpretation of probability.

In this case, what we want is to observe several independent repetitions of the experiment. After a while, some statistical regularity begins to emerge.

Frequency interpretation

Logically, if you run an experiment, and are interested in the probability of event A , then the events you are registering are A and A^c or *not* A .

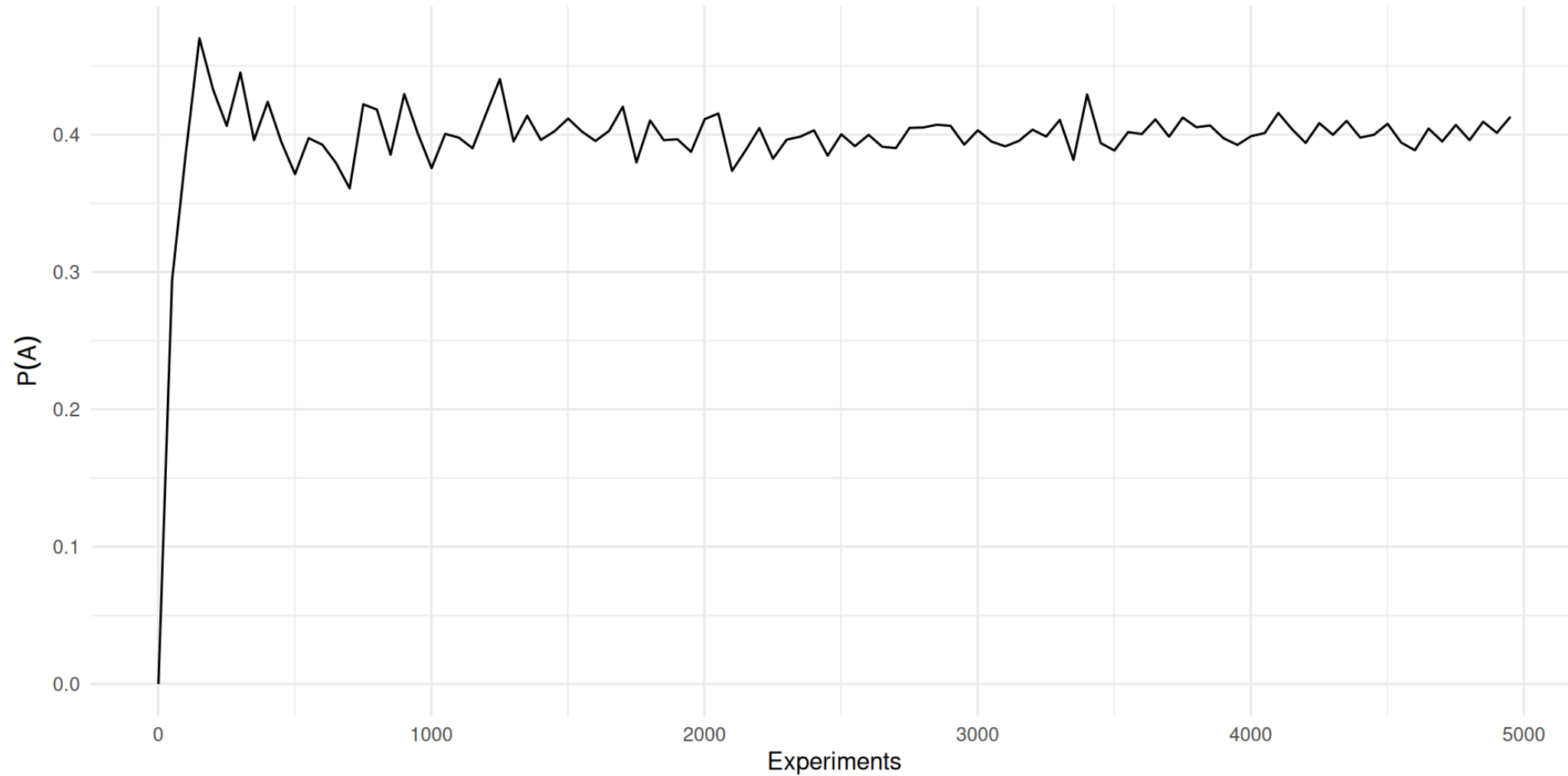
Every time you run your experiment, you count when you get an A and when you observe an A^c event. Obviously, the total number of experiments is how many times you observed A and how many times you observed A^c .

Example:

Experiment: Draw a random number in the interval $[0, 1]$. A denotes $x < 0.4$.

A	A^c	N	$P(A)$
0	1	1	0
2	8	10	0.2
17	33	50	0.34
39	61	100	0.39
217	283	500	0.434
802	1198	2000	0.401

Example:



Frequency interpretation

As you can see, the more experiments we run, the more stabilized the ratio of occurrences for A over the total number of experiments. More generally:

$$P(A) = \lim_{N \rightarrow \infty} \frac{A \text{ occurrences}}{N - \text{Number of Experiments}}$$

This is the relative frequency of A in N experiments: f_A

Not always possible to repeat that many times the experiment in the **same** conditions.

About our previous example

It seems that the probability that the random number between 0 and 1 is below 0.4 is approximately 40%. The more experiments we run, the closer our relative frequency is to that number.

$$P(A) \xrightarrow{N \rightarrow \infty} 0.4$$

By the way, we will see later that theoretically, indeed $P(A) = 0.4$

? The Concept of Probability · Question 1

You roll two fair dice. The probability that the sum equals 7 is:

A. $\frac{1}{12}$

B. $\frac{1}{6}$

C. $\frac{7}{36}$

D. $\frac{1}{9}$



B. There are 6 favourable outcomes out of 36:

$(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$, so $P = \frac{6}{36} = \frac{1}{6}$.

? The Concept of Probability · Question 2

Under the **frequency** interpretation, $P(A)$ is:

A. a subjective degree of belief

B. always $\frac{\#A}{\#\Omega}$

C. the limit of the relative frequency of A as the number of experiments grows

D. always 0.5

✓ **C.** The frequency interpretation repeats the experiment many times and takes $P(A) = \lim_{N \rightarrow \infty} \frac{\text{occurrences of } A}{N}$. The $\frac{\#A}{\#\Omega}$ formula is the *classical* definition instead, and it needs equally likely outcomes.



The Concept of Probability · Question 3

Roll two fair dice. Let A be “both dice show the same number” and B be “the sum is at least 10”.

Compute $P(A)$, $P(B)$ and $P(A \cap B)$.



The Concept of Probability · Solution

$$\#\Omega = 36.$$

$$A = \{(1, 1), \dots, (6, 6)\}, \text{ so } P(A) = \frac{6}{36} = \frac{1}{6}.$$

$$B = \{(4, 6), (5, 5), (6, 4), (5, 6), (6, 5), (6, 6)\}, \text{ so } P(B) = \frac{6}{36} = \frac{1}{6}.$$

$$A \cap B = \{(5, 5), (6, 6)\}, \text{ so } P(A \cap B) = \frac{2}{36} = \frac{1}{18}.$$

Practice

Practice

If there is time left, two quick ones. Difficulty goes  warm-up,  standard,  exam level,  stretch.

Practice · Exercise 1

Consider the following events:

A. “get a 7 when rolling a cubic die 🎲”

B. “Spain wins the next World Cup ⚽”

C. “rain in London ☔”

Which is correct?

1. A is impossible, while C is unlikely
 2. B is certain, and C is very likely
 3. $P(B)$ and $P(C)$ are computed using the subjective notion of probability
 4. A and C are impossible
- ✓ **3.** Nobody can repeat the next World Cup many times, nor list equally likely outcomes. What is left is a belief.

Practice · Exercise 2

When computing a probability from the analysis of all possible (equally likely) outcomes, we are using:

- A. the frequentist definition of probability
 - B. the classic definition of probability
 - C. the subjective definition of probability
 - D. the axiomatic definition of probability
-  **B.** This is Laplace: favourable cases over possible cases.



Homework

Problem set **2.1**, Questions 1 and 2, to check the ideas of today.