



Uniform, Exponential and Normal Distributions

Statistics I · Lecture 11

Today

- Uniform distribution
- Exponential distribution and its lack of memory
- Normal distribution: standard normal, standardization and the table
- Practice, from  to 

Disclaimer

These slides are a free translation and adaptation from the slide deck for Estatística I by Prof. Sandra Custódio and Prof. Teresa Ferreira from the Lisbon Accounting and Business School, Polytechnic University of Lisbon.

Uniform Distribution

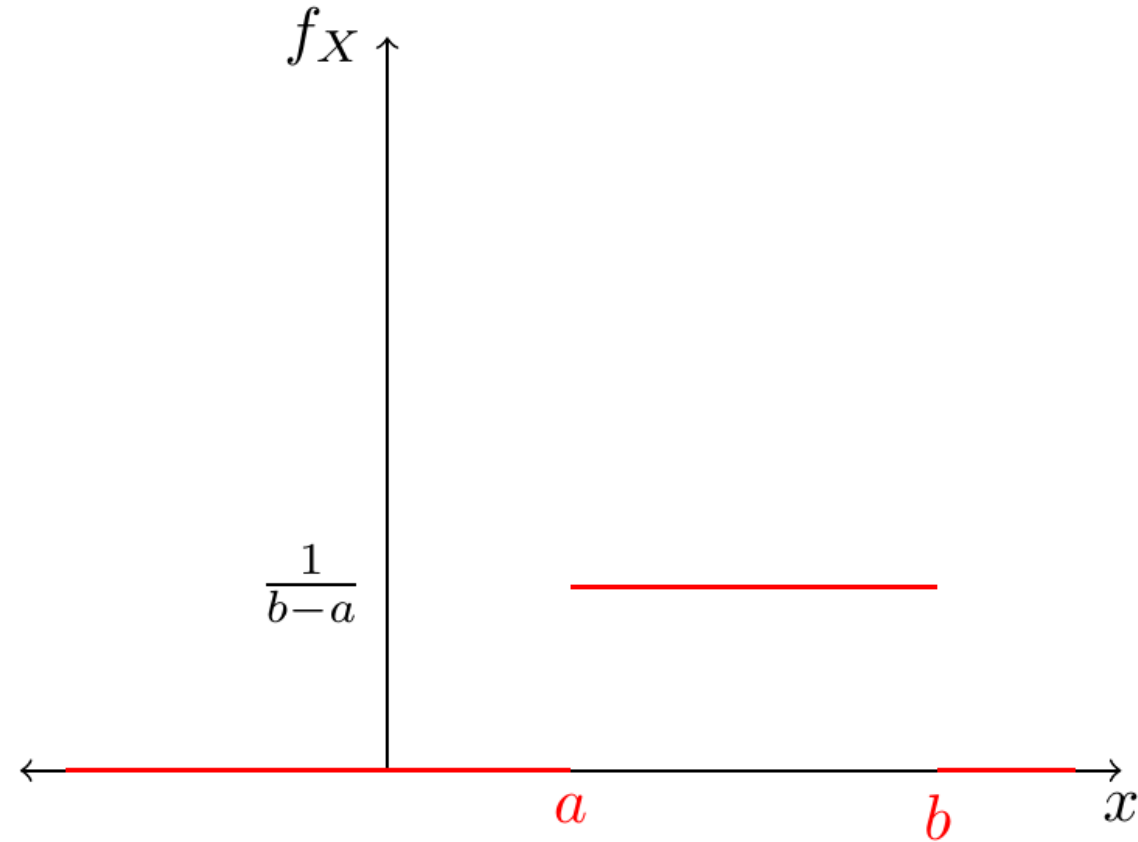
Continuous Uniform Distribution

A r.v. X follows a uniform distribution in $[a, b] \subset \mathbb{R}$ with $-\infty < a < b < \infty$, if its probability density is given by:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

We write $X \sim U[a, b]$

Continuous Uniform Distribution



Continuous Uniform Distribution

The cumulative distribution function is given by:

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x < b \\ 1 & x \geq b \end{cases}$$

Continuous Uniform Distribution

Note, this distribution is *symmetric*, and its first two moments are:

- $E[X] = \frac{a+b}{2}$
- $V[X] = \frac{(b-a)^2}{12}$

Continuous Uniform Distribution - Example

The length of small spots in a TV network is a r.v. X distributed $U[5, 12]$.

1. Find its distribution function.
2. What is the probability of a small spot last at least 7 seconds?
3. The probability that a small spot lasts more than 6 seconds, given it never lasts more than 10 seconds is?
4. Find $E[X]$ and $V[X]$

Solution

Continuous Uniform Distribution - Example

Let the r.v. X be distributed $U[2, b]$ with $b > 2$. What value must b take to make $P(3 \leq X \leq 5) = 0.4$?

Solution

? Uniform · Question 1

If $X \sim U[a, b]$, then $E[X]$ equals:

A. $\frac{a+b}{2}$

B. $\frac{b-a}{2}$

C. $\frac{1}{b-a}$

D. $\frac{(b-a)^2}{12}$

 **A.** The Uniform is symmetric, so its mean sits at the midpoint of the interval. $\frac{(b-a)^2}{12}$ is the variance.

? Uniform · Question 2

For $X \sim U[0, 10]$, the probability $P(X = 5)$ is:

A. 0.1

B. 1

C. 0.5

D. 0

☒ **D.** X is continuous, so every single point has probability zero, no matter how flat the density is.



Uniform · Question 3

Let $X \sim U[5, 15]$.

Compute $E[X]$, $V[X]$, $P(X \leq 8)$ and $P(7 < X \leq 12)$.



Uniform · Solution

$$E[X] = \frac{5+15}{2} = 10 \cdot V[X] = \frac{(15-5)^2}{12} = \frac{100}{12} \approx 8.33$$

$$F_X(x) = \frac{x-5}{10} \text{ on } [5, 15], \text{ so } P(X \leq 8) = \frac{3}{10} = 0.3$$

$$P(7 < X \leq 12) = \frac{12-7}{10} = 0.5$$

Exponential Distribution

Exponential Distribution

The exponential distribution is rooted in the Poisson distribution, reflecting the waiting time between events originated according to a Poisson process.

Nevertheless, we can apply the exponential distribution to many other phenomena.

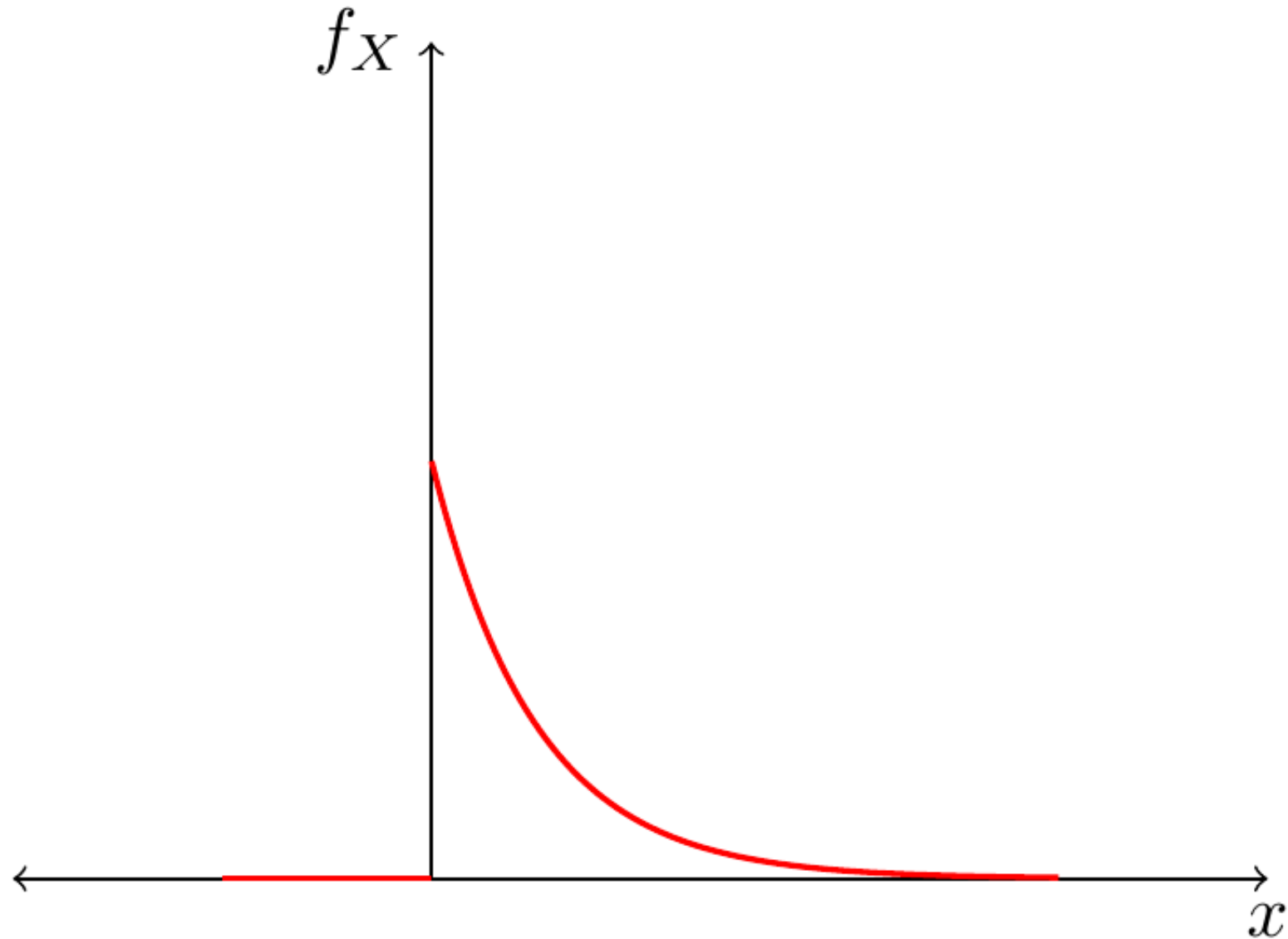
Exponential Distribution

A r.v X is distributed exponentially, with parameter $\lambda > 0$, $X \sim \text{Exp}(\lambda)$, if its probability density function is given by:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

λ is the *rate*: the expected number of events per unit of time (or space). The expected waiting time between events is $\frac{1}{\lambda}$.

Exponential Distribution



Exponential Distribution

The cumulative probability function is:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

And its moments are:

- $E[X] = \frac{1}{\lambda}$
- $V[X] = \frac{1}{\lambda^2}$

Exponential Distribution

Property

Lack of memory of the exponential distribution:

Let the r.v. $X \sim \text{Exp}(\lambda)$, then:

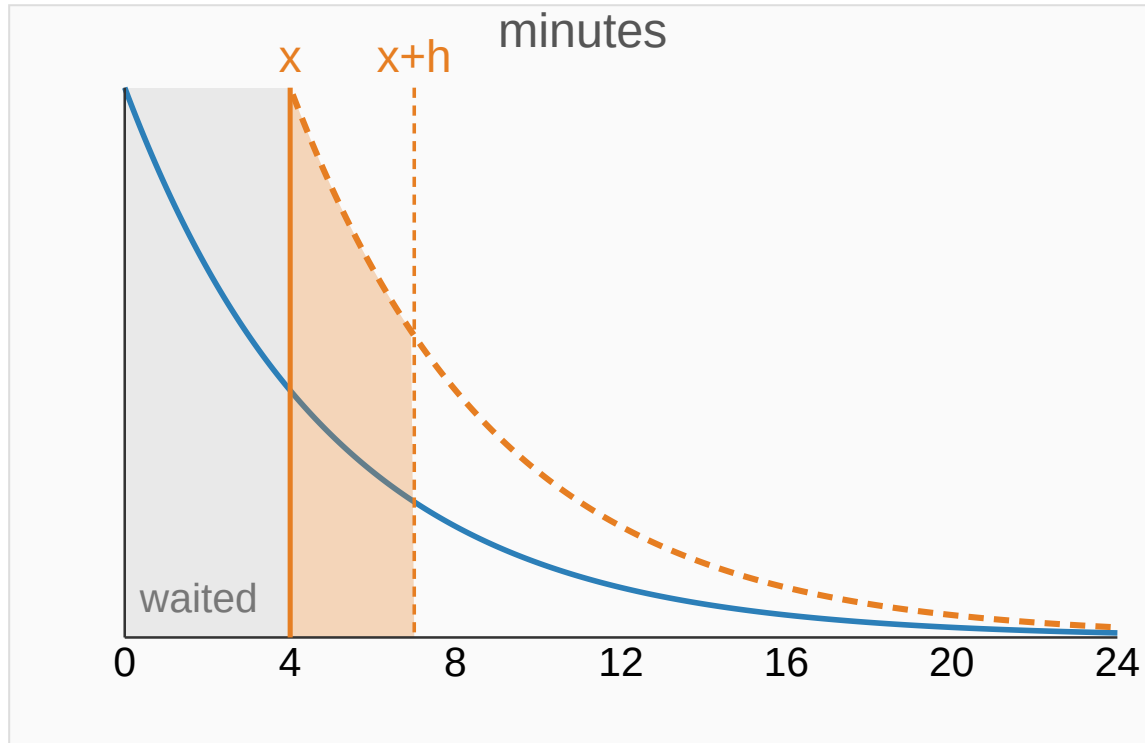
$$P(X > x + h | X > x) = P(X > h)$$

with $x, h > 0$

Considering the survival application of this distribution, this property states that, the time left to live is independent of what it already lived.

Does waiting change anything?

Grey = time you have **already** waited with nothing happening. Blue = the original distribution from time zero. Orange dashed = the distribution of what is **still to come**.



already waited $x = 4.00$



extra wait $h = 3.00$



$$P(X > 7.00 \mid X > 4.00) = \mathbf{0.5488}$$

$$P(X > 3.00) = \mathbf{0.5488}$$

identical, for every x

The orange curve is the blue one slid to start at x . Its shape never changes, so the expected remaining wait is still 5 minutes.

What the chart showed

Drag x anywhere you like. The orange curve slides along with it, but its **shape never changes**: it is the blue curve translated, not a flatter or steeper one. And the two probabilities in the panel stay equal to four decimals.

Algebraically: $P(X > x + h \mid X > x) = \frac{e^{-\lambda(x+h)}}{e^{-\lambda x}} = e^{-\lambda h} = P(X > h)$. The $e^{-\lambda x}$ cancels, and with it everything the past could have told you.

A component that has run 10 years is exactly as likely to survive another year as a brand new one. That is often the wrong model for **wear**, which is precisely why you must check whether the exponential fits before using it.

Exponential Distribution - Example

In a factory, the execution time of a piece is random variable distributed exponentially with expected value of 5 minutes.

1. For this distribution, what is the value of λ ?
2. You know that the piece has been in execution by at least 2 minutes.
What is the probability that 4 more minutes are necessary to complete the piece?
3. Picking randomly 5 pieces. What is the probability that 2 out of the 5 were produced in less than 4 minutes?

Solution

Exponential Distribution - Example

The time it takes until the first consultation, and between consultations, in the clinic of Dr. Shawn are independent and distributed exponentially with $\lambda = 0.1$.

What is the probability that no consultation occurs before the first 10 minutes?

Solution

? Exponential · Question 1

Buses arrive following an Exponential model with $\lambda = 2$ per hour. The expected waiting time for the next bus is:

A. 2 hours

B. 0.5 hours

C. 4 hours

D. 1 hour

✓ **B.** λ is a **rate**, here 2 buses per hour. The expected waiting time is $\frac{1}{\lambda} = 0.5$ hours, that is 30 minutes.

? Exponential · Question 2

Memorylessness says that $P(X > x + h \mid X > x)$ equals:

A. $P(X > h)$

B. $P(X > x)$

C. $P(X > x + h)$

D. 0

✓ A. Having already waited x tells you nothing. The clock restarts, so only the extra h matters.



Exponential · Question 3

Buses arrive with $\lambda = 0.05$ per minute.

Compute the expected waiting time, $P(X > 10)$, and $P(X > 25 \mid X > 15)$.



Exponential · Solution

$$E[X] = \frac{1}{0.05} = 20 \text{ minutes.}$$

$$P(X > 10) = 1 - F(10) = e^{-0.05 \times 10} = e^{-0.5} \approx 0.6065$$

By memorylessness, $P(X > 25 | X > 15) = P(X > 10) \approx 0.6065$, the same number. 

Normal Distribution

Normal Distribution

The **Gaussian** or **Normal** Distributions is one of the most used distributions, playing a key role in statistical inference.

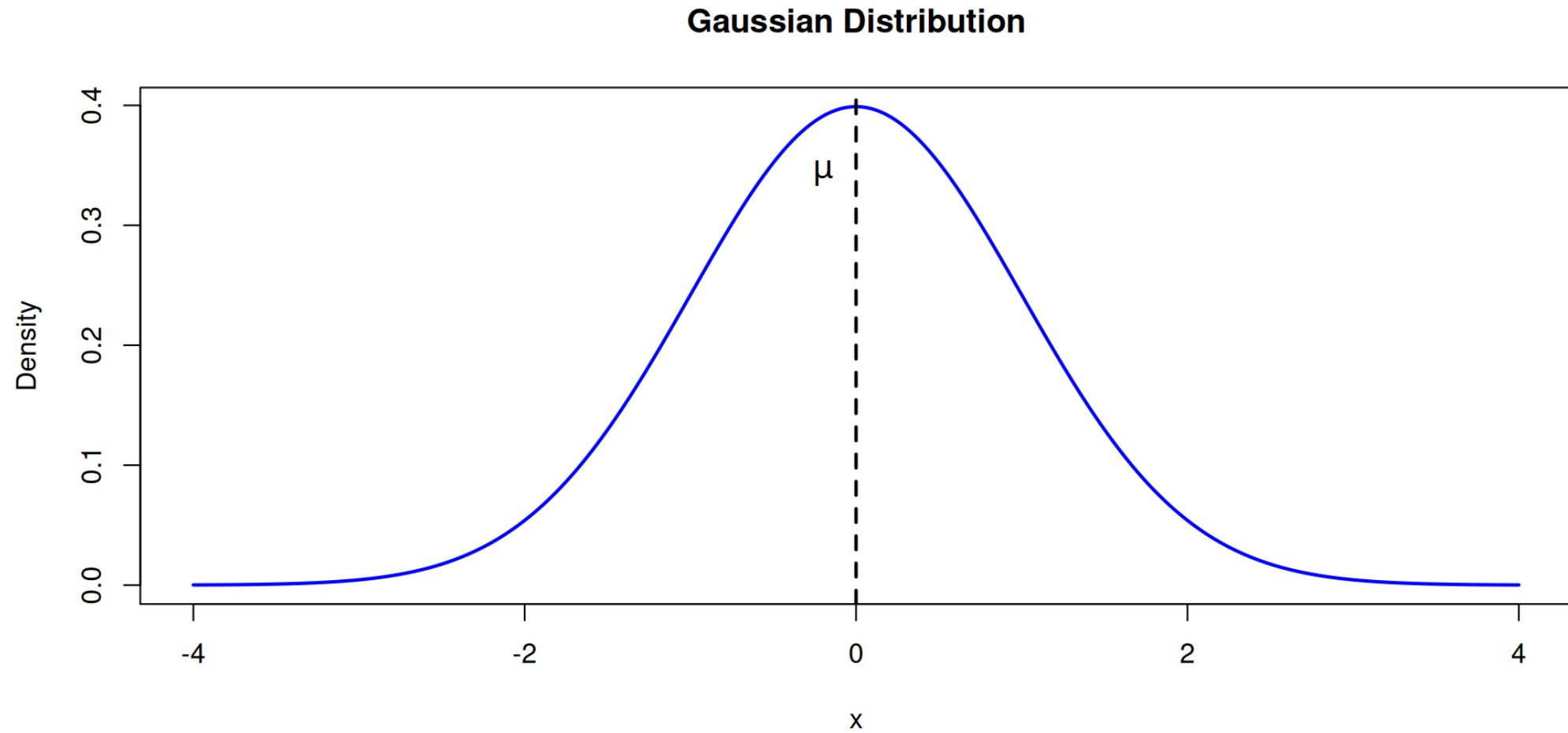
The r.v. X is normally distributed, $X \sim N(\mu, \sigma^2)$, if its density and cumulative probability distribution functions:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$P(X \leq a) = F(a) = \int_{-\infty}^a f(t)dt$$

Where $E[X] = \mu$ and $Var[X] = \sigma^2$

Normal Distribution



Standard Normal

If $\mu = 0$ and $\sigma^2 = 1$ we call this the *standard* normal distribution, where $Z \sim N(0, 1)$.

! Theorem

Let $X \sim N(\mu, \sigma^2)$. Let $Z = \frac{X - \mu}{\sigma}$, then

$$Z \sim N(0, 1)$$

Standard Normal

That is, we can *standardize* a r.v. distributed Normally. Its probability function is denoted Φ , and its density is ϕ .

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

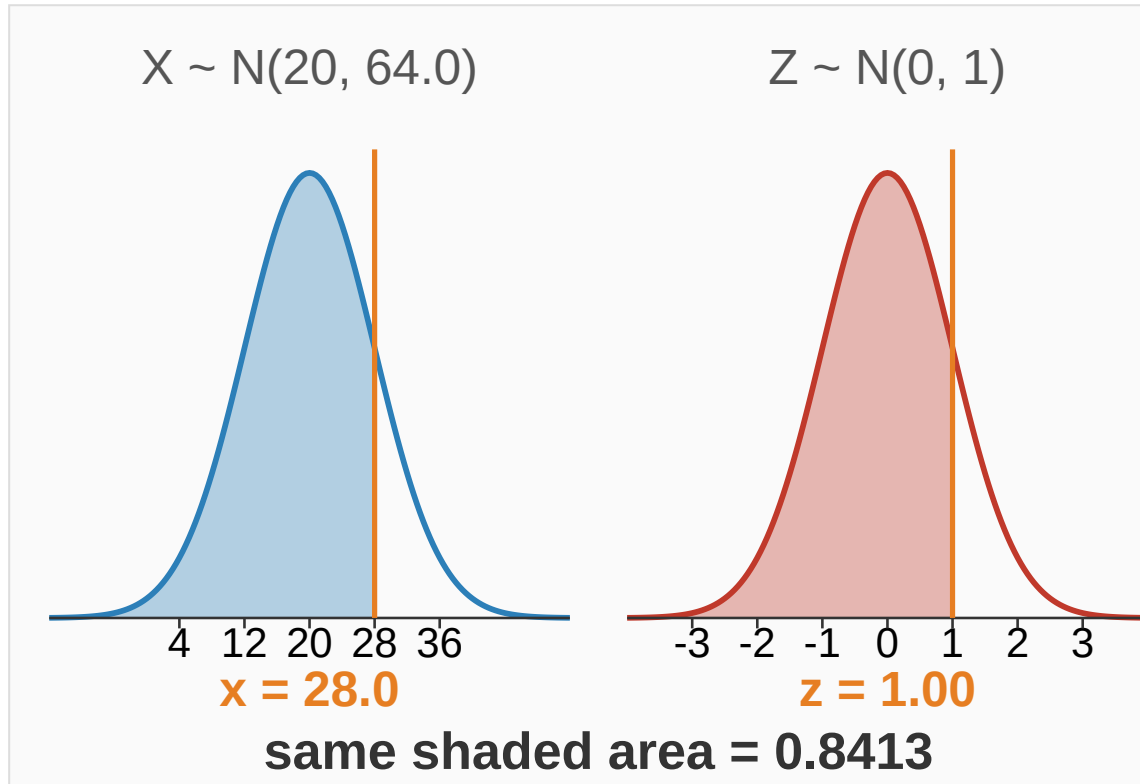
Standard Normal

How do we know if a r.v. follows a Normal distribution?

1. There is no rule, you need to look at the data (histogram) and decide.
2. This distribution is popular because of the Law of Large Numbers and the Central Limit Theorem (covered in Statistics II)
3. Even though it is by far the most popular, not always it is used correctly.

Standardizing, seen side by side

Left: your $X \sim N(\mu, \sigma^2)$. Right: the same problem after standardizing. **The two shaded areas are always equal**, which is why one table serves every Normal.



$\mu = 20$



$\sigma = 8.0$



cut-off $x = 28.0$



$$z = (28.0 - 20) / 8.0 = 1.00$$

$$\Phi(1.00) = 0.8413$$

$$P(X \leq 28.0) = 0.8413$$

$$P(X > 28.0) = 0.1587$$

Look up 1.00 in the table: row 1.0, column
.00

What the chart showed

Change μ and the left curve slides; change σ and it widens or narrows. The **right curve never moves**: standardizing has already absorbed both.

The orange cut-off tracks the same point in both pictures, at x on the left and at $z = \frac{x - \mu}{\sigma}$ on the right, and the shaded areas stay equal to the last decimal.

This is why one page of tabulated $\Phi(z)$ is enough for every Normal you will ever meet. You never tabulate X , you convert it. Note the readout tells you the exact row and column to look up.

Example 1

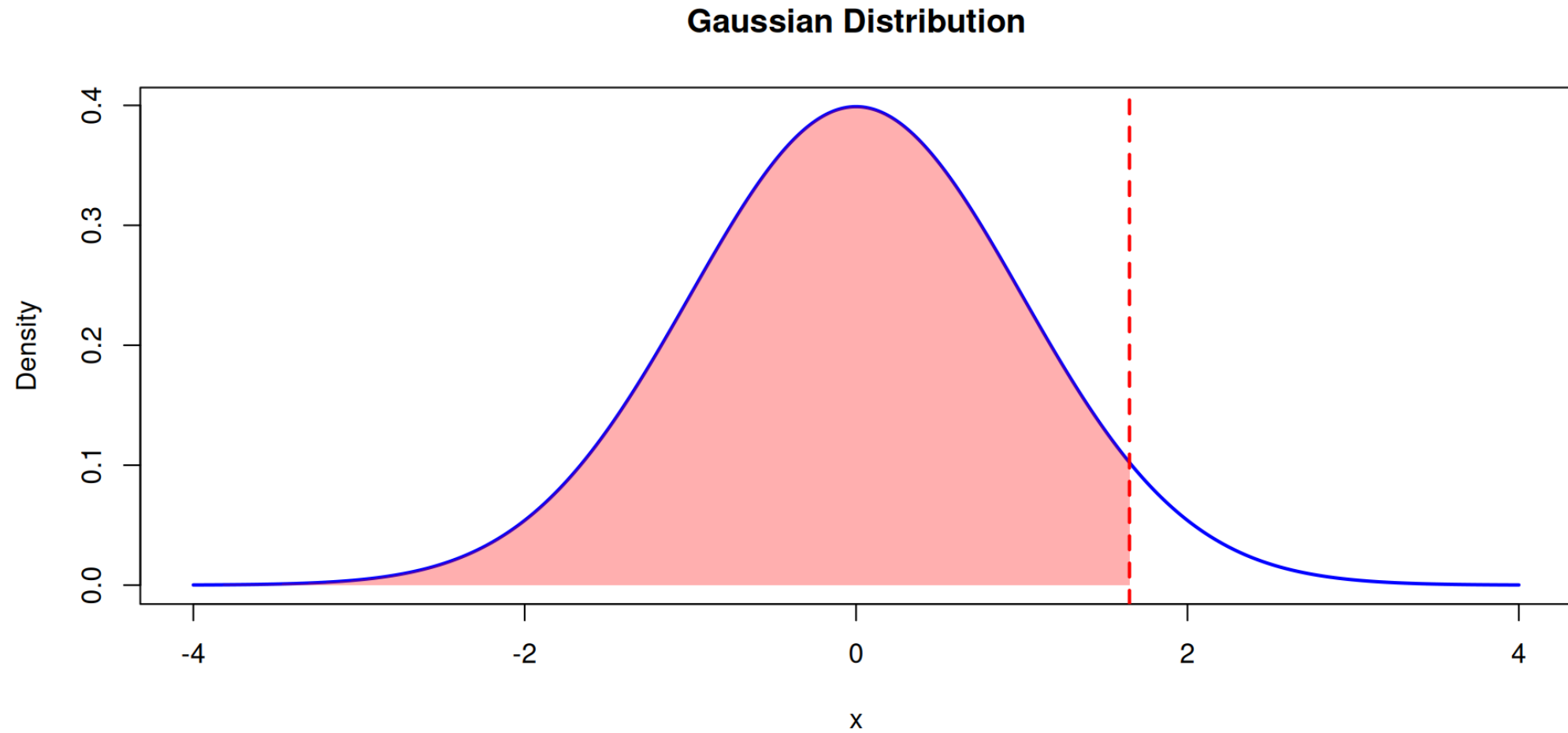
Consider the r.v. $Z \sim N(0, 1)$. Find $P(Z \leq 1.65)$.

$$P(Z \leq 1.65) = F(1.65) = \Phi(1.65) = 0.9505$$

Warning

1. This cannot be solved using the classical integration. Other techniques (approximations) are necessary.
2. We will rely on Tables to find out the values for the Standard Normal distribution. Remember you can always *standardize* other Normal random variables.
3. If $z < 0$, recall that the distribution is symmetric, then $\Phi(-z) = 1 - \Phi(z)$

Example 1



Example 2

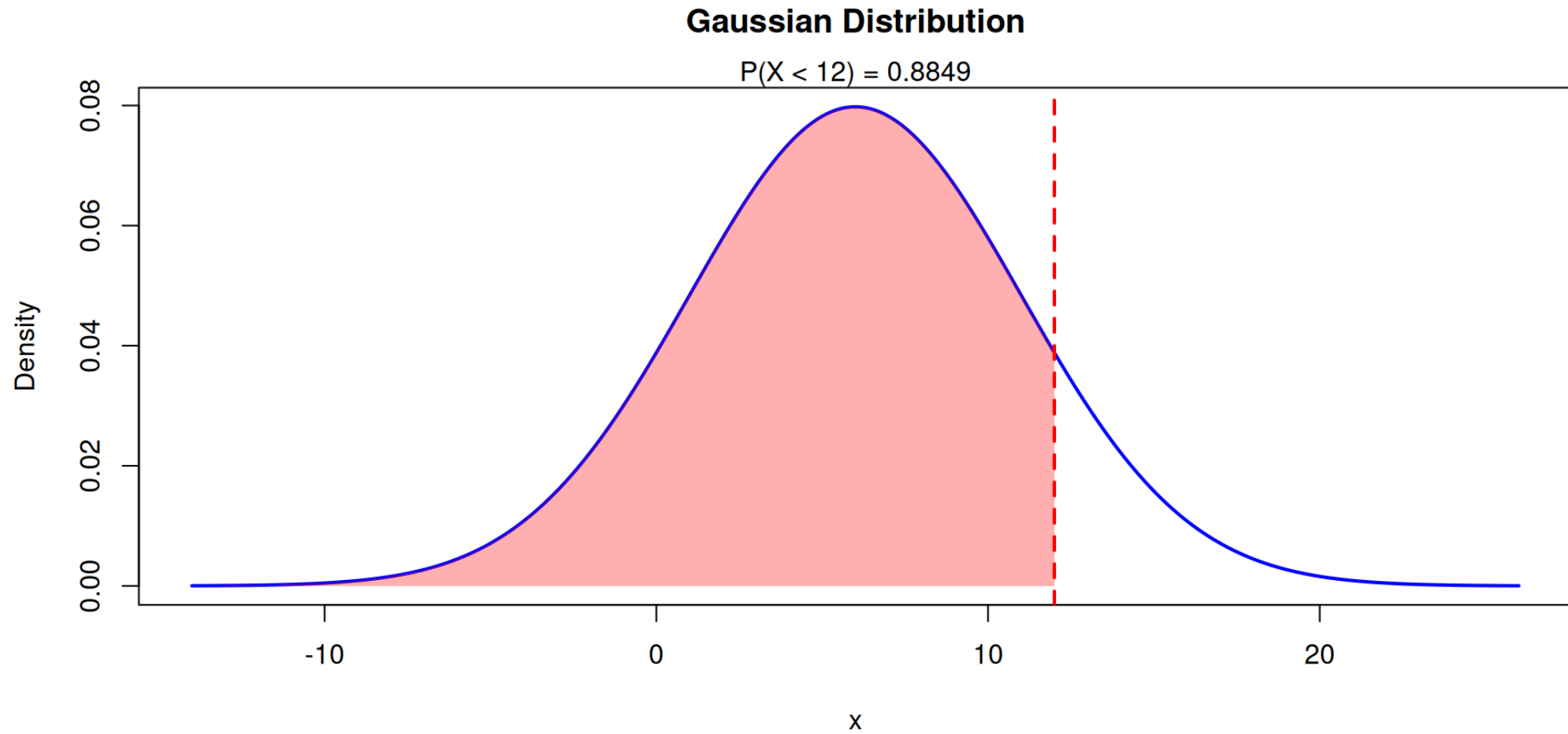
Consider the r.v. $X \sim N(6, 25)$. Find $P(X \leq 12)$.

$$P(X \leq 12) = P\left(\frac{X - 6}{5} \leq \frac{12 - 6}{5}\right)$$

Or

$$P(Z \leq 1.2) = \Phi(1.2) = 0.8849$$

Example 2



Practice

Practice

The exercises go from easy to hard:

-  warm-up: one formula, one step
-  standard: two or three steps
-  exam level: this is what you will find in the midterm
-  stretch: you need to combine several things

Keep the [Normal table](#) open.

Practice · Exercise 1

Let $X \sim \text{Exp}(0.5)$. Find $P(X > 2)$.

$$P(X > 2) = 1 - F(2) = e^{-0.5 \times 2} = e^{-1} \approx 0.3679$$

Practice · Exercise 2

Let $Z \sim N(0, 1)$. Find:

(a) $P(0 < Z \leq 2.05)$ **(b)** $P(-1.22 \leq Z < 1.05)$ **(c)** $P(Z \geq -2.05)$

(d) k such that $P(|Z| > k) = 0.05$ **(e)** k such that $P(|Z| < k) = 0.90$

(a) $\Phi(2.05) - \Phi(0) = 0.9798 - 0.5 = 0.4798$

(b) $\Phi(1.05) - \Phi(-1.22) = 0.8531 - (1 - 0.8888) = 0.7419$

(c) By symmetry, $P(Z \geq -2.05) = P(Z \leq 2.05) = 0.9798$

(d) 0.025 in each tail, $\Phi(k) = 0.975 \Rightarrow k = 1.96$ **(e)** 0.05 in each tail,
 $\Phi(k) = 0.95 \Rightarrow k \approx 1.645$

Practice · Exercise 3

The number of cars 🚗 checked in an inspection center, per hour, is distributed Poisson with an expected value of 4.

- (a) What is the expected time between two cars?
- (b) Find the probability of waiting more than 30 minutes for the next car.

If the number of events per hour is $Poi(4)$, the time between events, in hours, is $T \sim Exp(4)$.

- (a) $E[T] = \frac{1}{4}$ of an hour, 15 minutes.
- (b) $P(T > 0.5) = e^{-4 \times 0.5} = e^{-2} \approx 0.1353$. Or, with the Poisson: no cars in 30 minutes, $P(Y = 0)$ with $Y \sim Poi(2)$. Same number!

Practice · Exercise 4

The bus 🚌 arrives at your stop at a random moment between 0 and 20 minutes from now, all moments equally likely. Let W be your waiting time.

(a) Find $P(W > 15)$.

(b) You have waited 10 minutes already. Find $P(W > 15 | W > 10)$.

(c) Find $E[W]$ and $V[W]$.

$$W \sim U(0, 20)$$

$$\textbf{(a)} \ P(W > 15) = \frac{20-15}{20} = 0.25 \quad \textbf{(b)}$$

$$P(W > 15 | W > 10) = \frac{P(W > 15)}{P(W > 10)} = \frac{0.25}{0.5} = 0.5$$

$$\textbf{(c)} \ E[W] = \frac{0+20}{2} = 10 \text{ and } V[W] = \frac{(20-0)^2}{12} \approx 33.33$$

Practice · Exercise 5

The lifetime of a light bulb , in hours, is distributed exponentially with mean 1000 hours.

- (a)** Find the probability that a bulb lasts more than 1500 hours.
- (b)** Find the median lifetime. Is it above or below the mean?
- (c)** A box has 4 bulbs, independent. Find the probability that at least one of them lasts more than 1500 hours.



Practice · Exercise 5 · Solution

$$T \sim \text{Exp}\left(\frac{1}{1000}\right)$$

$$\text{(a)} \quad P(T > 1500) = e^{-1500/1000} = e^{-1.5} \approx 0.2231$$

(b) $F(m) = 1 - e^{-m/1000} = 0.5 \Rightarrow m = 1000 \ln 2 \approx 693$ hours. Below the mean: a few bulbs last very long and pull the mean up.

(c) $N \sim \text{Bin}(4, 0.2231)$ counts the bulbs lasting more than 1500 hours:
 $P(N \geq 1) = 1 - 0.7769^4 \approx 0.6358$

● Practice · Exercise 6

Go back to the bus 🚌 of Exercise 4. Now suppose the waiting time is instead $W \sim \text{Exp}(0.1)$, so it has the same mean, 10 minutes.

(a) Find $P(W > 15)$ and $P(W > 15 | W > 10)$.

(b) Compare with Exercise 4. With which bus is it worth to keep waiting?

(a) $P(W > 15) = e^{-1.5} \approx 0.2231$ and, by the lack of memory,
 $P(W > 15 | W > 10) = P(W > 5) = e^{-0.5} \approx 0.6065$

(b) With the uniform bus, having waited 10 minutes makes it more likely the bus arrives soon, it must arrive before minute 20. With the exponential bus, the 10 minutes you already waited are worth nothing 😞, you are back at the start.



Homework

Problem set **3.2**, Questions 1 to 7.

References

- Murteira, B.; Silva Ribeiro, C.; Andrade e Silva, J. & Pimenta, C., Introdução à Estatística, Escolar Editora, McGraw-Hill, 2010
- Paulino, C. D. & Branco, J. A. (2005). Exercícios de Probabilidade e Estatística. Escolar Editora
- Pimenta, F., Andrade e Silva, J.; Silva Ribeiro, C. & Murteira, B., Introdução à Estatística, 3ª Edição, Escolar Editora, 2015
- Ferreira, T., Custódio, S.G., Modelos Probabilísticos, Síntese Teórica e Exercícios Resolvidos, Edições Sílabo (1ª Edição), 2023

Appendix

Standard Normal Table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621

Uniform Exercise

$$1. \quad F_X(x) = \begin{cases} 0 & x < 5 \\ \frac{x-5}{12-5} & 5 \leq x < 12 \\ 1 & x \geq 12 \end{cases}$$

$$2. \quad F_X(12) - F_X(7) = 1 - \frac{2}{7} = \frac{5}{7}$$

$$3. \quad \frac{F_X(10) - F_X(6)}{F_X(10)} = \frac{5/7 - 1/7}{5/7} = 4/5$$

$$4. \quad E[X] = \frac{5+12}{2} = 8.5, \quad V[X] = \frac{(12-5)^2}{12} = \frac{49}{12} \approx 4 \Rightarrow \sigma_X \approx \sqrt{4} \approx 2$$

[Back to the exercise](#)

Uniform Exercise

$$1. P(3 \leq X \leq 5) = 0.4 \Rightarrow F_X(5) - F_X(3) = 0.4$$

$$2. \frac{5-2}{b-2} - \frac{3-2}{b-2} = 0.4 \Rightarrow \frac{3}{b-2} - \frac{1}{b-2} = 0.4$$

$$3. \frac{2}{b-2} = 0.4 \Rightarrow \frac{1}{b-2} = 0.2, \text{ but } 0.2 = \frac{2}{10} = \frac{1}{5}$$

$$4. \frac{1}{b-2} = \frac{1}{5} \Rightarrow 5 = b - 2 \Rightarrow b = 7$$

Back to the exercise

Exponential Exercise

1. $\lambda = \frac{1}{5} = 0.2$. On average 0.2 pieces are produced per minute, that is one piece every 5 minutes.

$$X \sim \text{Exp}(0.2)$$

$$2. P(X > 6 | X > 2) = P(X > 4) = 1 - P(T < 4) = 1 - (1 - e^{-0.2 \times 4}) = e^{-0.8} \approx 0.4493$$

3. This problem calls for the binomial distribution, $N \sim \text{Bin}(n = 5, p)$, however, we do not know p . Here p is the probability that a single piece is produced in 4 minutes or less: $p = P(X < 4)$ with $X \sim \text{Exp}(0.2)$, so $p = 1 - e^{-0.2 \times 4} \approx 0.55$. Then $N \sim \text{Bin}(5, 0.55)$, and $P(N = 2) \approx 0.2757$.

[Back to the exercise](#)

Exponential Exercise

In this case the r.v. X is the time until the first consultation, and $X \sim \text{Exp}(0.1)$.

$$P(X > 10) = 1 - P(X < 10) = 1 - F(10)$$

Or

$$1 - (1 - e^{-0.1 \times 10}) = e^{-1} \approx 0.3679$$

[Back to the exercise](#)