



# The Normal Distribution and Midterm 2 Review

Statistics I · Lecture 12

# Today

- Normal distribution: worked examples
- Additivity and its corollaries
- Practice, and a mock test for Midterm 2

# Normal Distribution (continued)

## Example 3

If  $X \sim N(6, 25)$ , find  $P(6 < X \leq 12)$

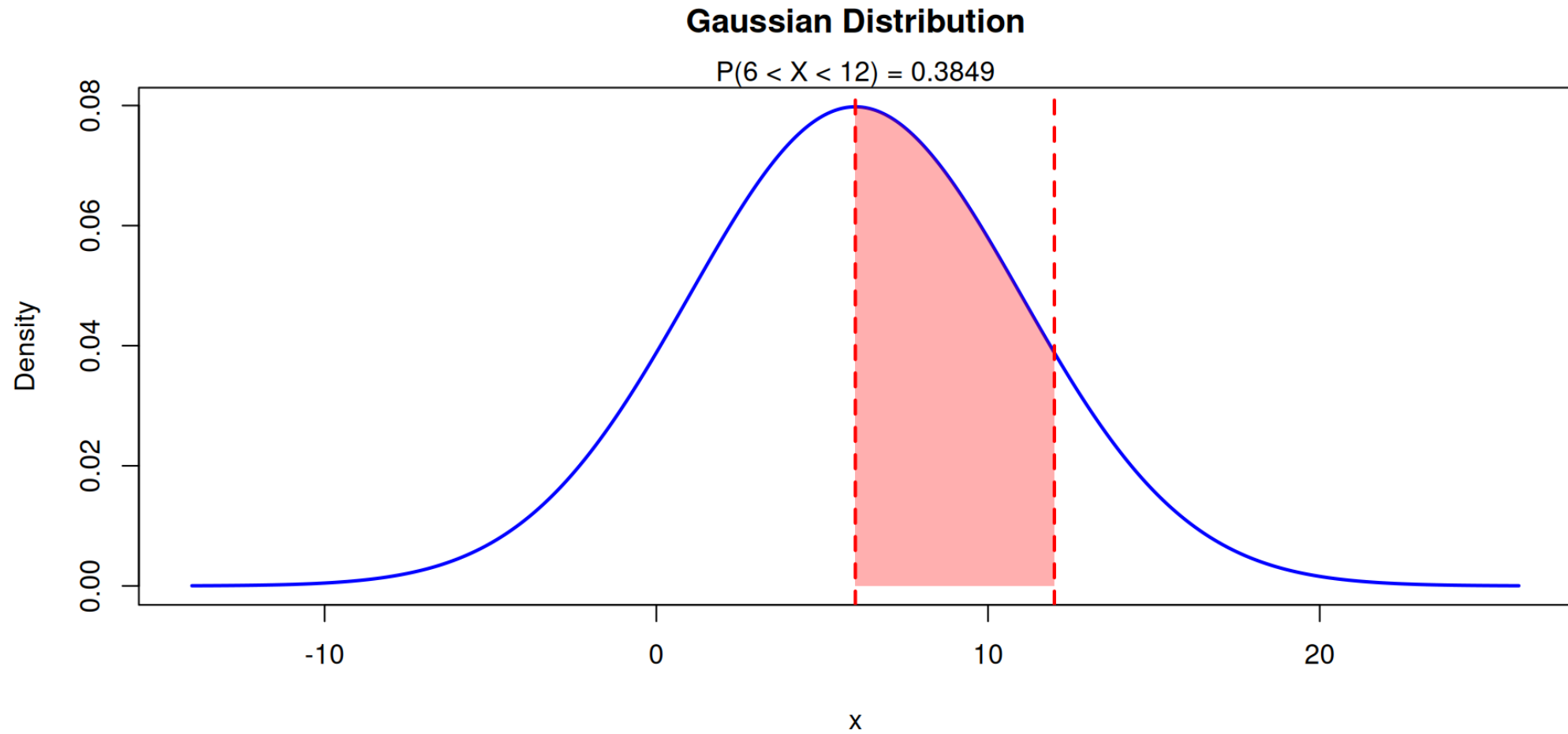
$$P(6 < X \leq 12) = F(12) - F(6) =$$

$$P\left(\frac{6 - 6}{5} < \frac{X - 6}{5} \leq \frac{12 - 6}{5}\right)$$

Or

$$\begin{aligned} P(0 < Z \leq 1.2) &= \Phi(1.2) - \Phi(0) \\ &= 0.8849 - 0.5 = 0.3849 \end{aligned}$$

# Example 3

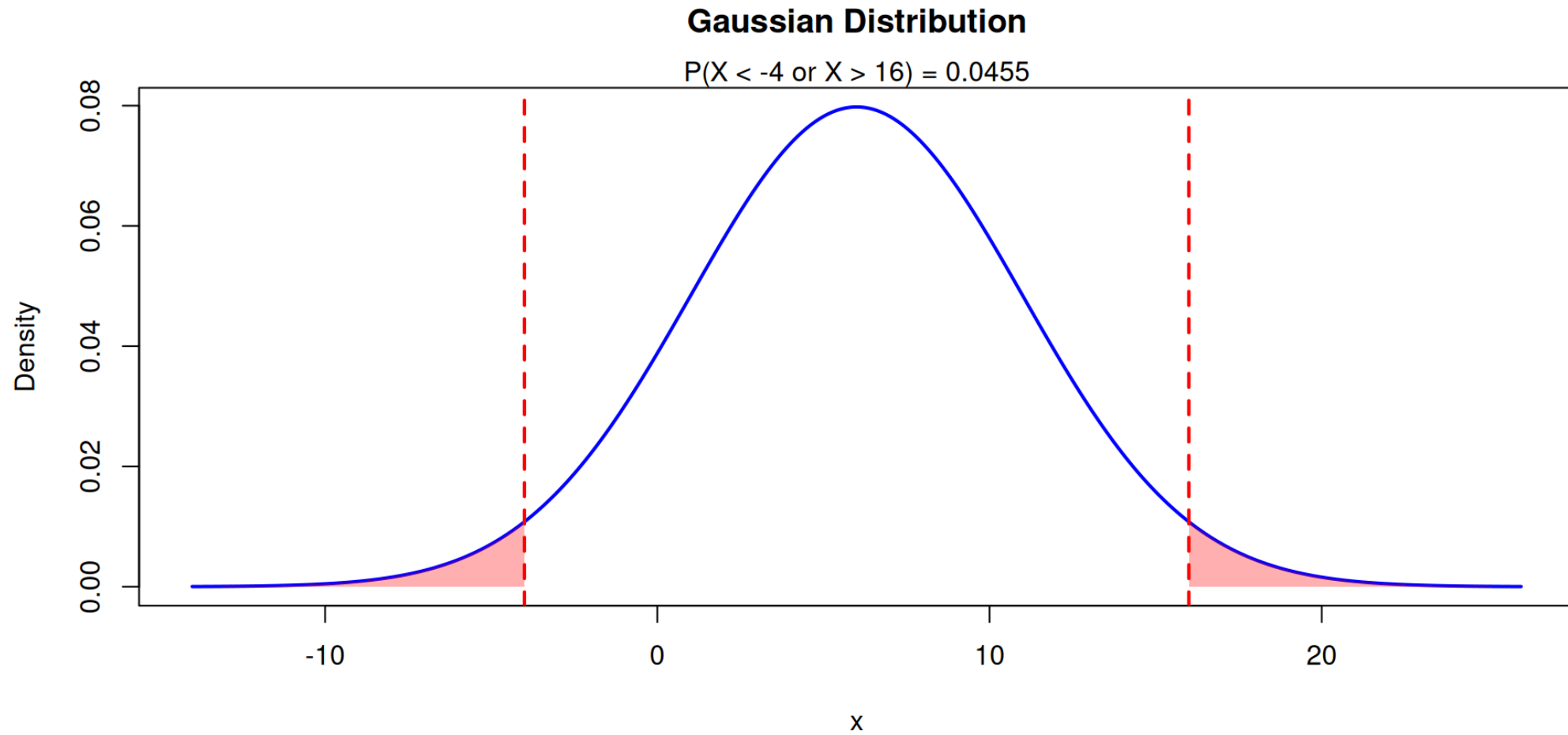


## Example 4

Let  $X \sim N(6, 25)$ . Find  $P(X \leq -4)$  and  $P(X \geq 16)$

$$\begin{aligned} P(X \leq -4) &= P\left(\frac{X - 6}{5} \leq \frac{-4 - 6}{5}\right) = \\ P(Z \leq -2) &= \Phi(-2) = 1 - \Phi(2) = 0.0228 \end{aligned}$$

# Example 4



## Example 5

Check that  $P(X \leq -4) = P(X \geq 16)$

Because of symmetry:  $P(X \leq \mu - k) = P(X \geq \mu + k) \forall k \in \mathbb{R}$

$$\begin{aligned} P(X \leq -4) &= P(X \leq 6 - 10) = \\ &= P(X \geq 6 + 10) = P(X \geq 16) \end{aligned}$$



### Corollary

$$\Phi(-k) = P(Z \leq -k) = P(Z \geq k) = 1 - P(Z \leq k) = 1 - \Phi(k)$$

## Example 6

Consider the distribution  $X \sim N(6, 25)$

Find  $P(0 \leq X \leq 8)$

$P(0 \leq X \leq 8) = P(0 < X \leq 8)$  which is equivalent to

$$\begin{aligned} P(-1.2 < Z \leq 0.4) &= \Phi(0.4) - \Phi(-1.2) = \\ &= \Phi(0.4) - [1 - \Phi(1.2)] = \\ &= 0.6554 - [1 - 0.8849] = 0.5403 \end{aligned}$$

## Example 7

Consider the r.v.  $X \sim N(6, 25)$ . Find  $P(|X - 6| > 10)$

$$P(|X - 6| > 10) = 1 - P(|X - 6| \leq 10)$$

$$1 - P(-10 \leq X - 6 \leq 10) = 1 - P(-2 < Z \leq 2)$$

$$1 - [\Phi(2) - \Phi(-2)] = 1 - [\Phi(2) - 1 + \Phi(2)]$$

$$2 - 2\Phi(2) = 2 - 2 \times 0.9772 = 0.0456$$

## Example 8

Consider the r.v.  $X \sim N(6, 25)$ , find  $k$  such that  $P(X > k) = 0.9$

$$P(X > k) = 0.9 \Leftrightarrow 1 - P(X \leq k) = 0.9$$

Or

$$P(X \leq k) = 0.10 \Leftrightarrow P\left(Z \leq \frac{k - 6}{5}\right) = 0.10$$

$$P\left(Z \leq \frac{k - 6}{5}\right) = 0.10 \Leftrightarrow \Phi\left(\frac{k - 6}{5}\right) = 0.10$$

Because of symmetry we have  $P(Z \leq -z) = P(Z \geq z)$

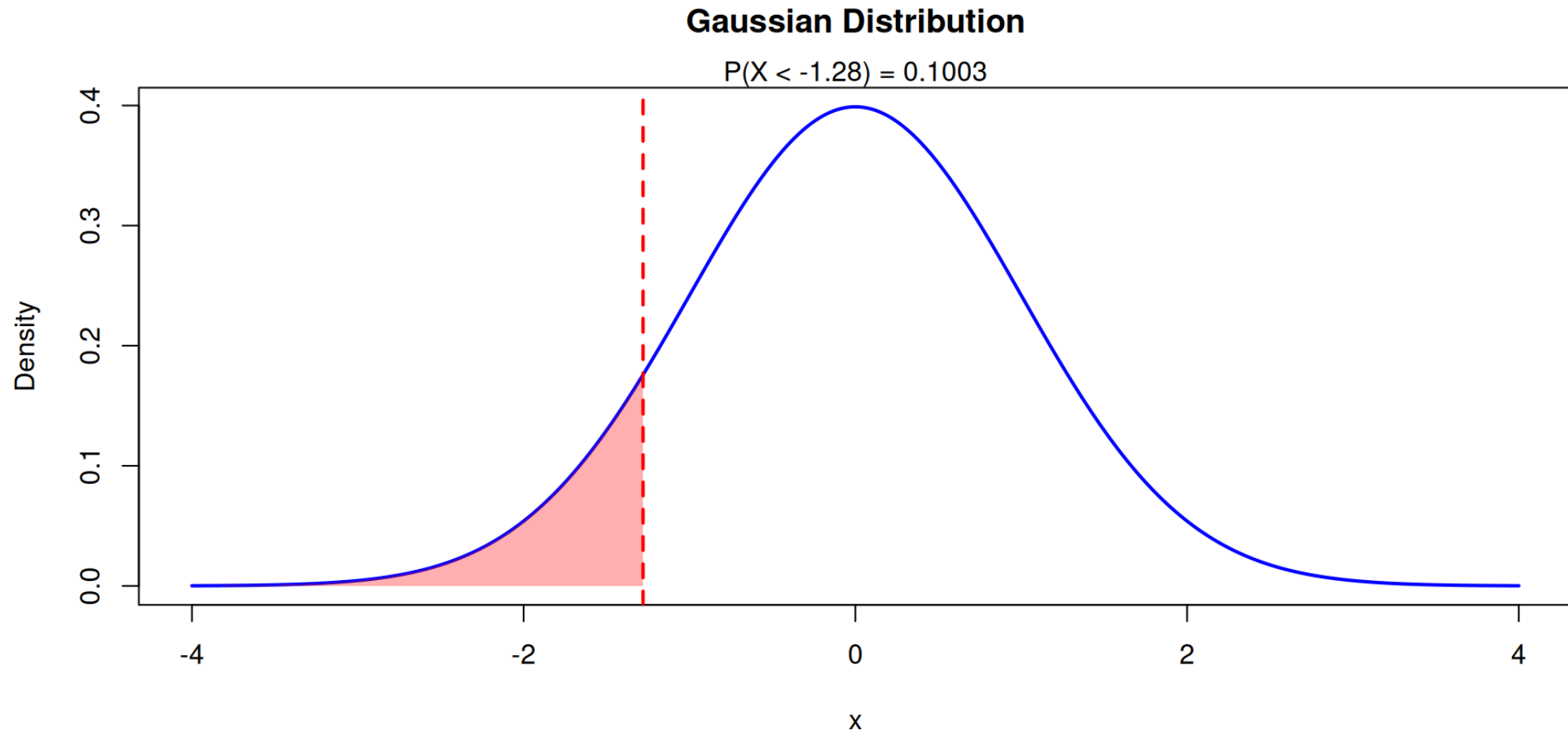
## Example 8

Using the table,  $P(Z \geq z) = 0.1$  means  $\Phi(z) = 0.9$ , so  $z = \Phi^{-1}(0.9) \approx 1.282$ , and by symmetry  $\Phi^{-1}(0.1) \approx -1.282$ . Substituting we get:

$$\Phi(-1.282) = 0.1 \Leftrightarrow \frac{k - 6}{5} = -1.282$$

$$k = 6 - 1.282 \times 5 = -0.41$$

# Example 8



# Normal Distribution

## ! Theorem: Normal additivity

If  $X_1 \sim N(\mu_1, \sigma_1^2)$  and  $X_2 \sim N(\mu_2, \sigma_2^2)$  are **independent**, then for any  $a, b \in \mathbb{R}$  we have that  $T = aX_1 + bX_2$ , where

$$T \sim N(\mu_T, \sigma_T^2)$$

To find  $\mu_T$  and  $\sigma_T^2$  remember the properties of the mean and variance.

## Example 9

Let the r.v.s  $X \sim N(6, 4)$  and  $Y \sim N(6, 4)$ , with  $T = 0.5X - Y$  Find  $\mu_T$  and  $\sigma_T^2$

$$\begin{aligned}\mu_T &= E[0.5X - Y] = 0.5E[X] - E[Y] = \\ &0.5 \times 6 - 6 = -3\end{aligned}$$

$$\begin{aligned}\sigma_T^2 &= V[0.5X - Y] = V[0.5X] + V[-Y] = \\ &0.5^2 V[X] + V[Y] = 0.25 \times 4 + 4 = 5\end{aligned}$$

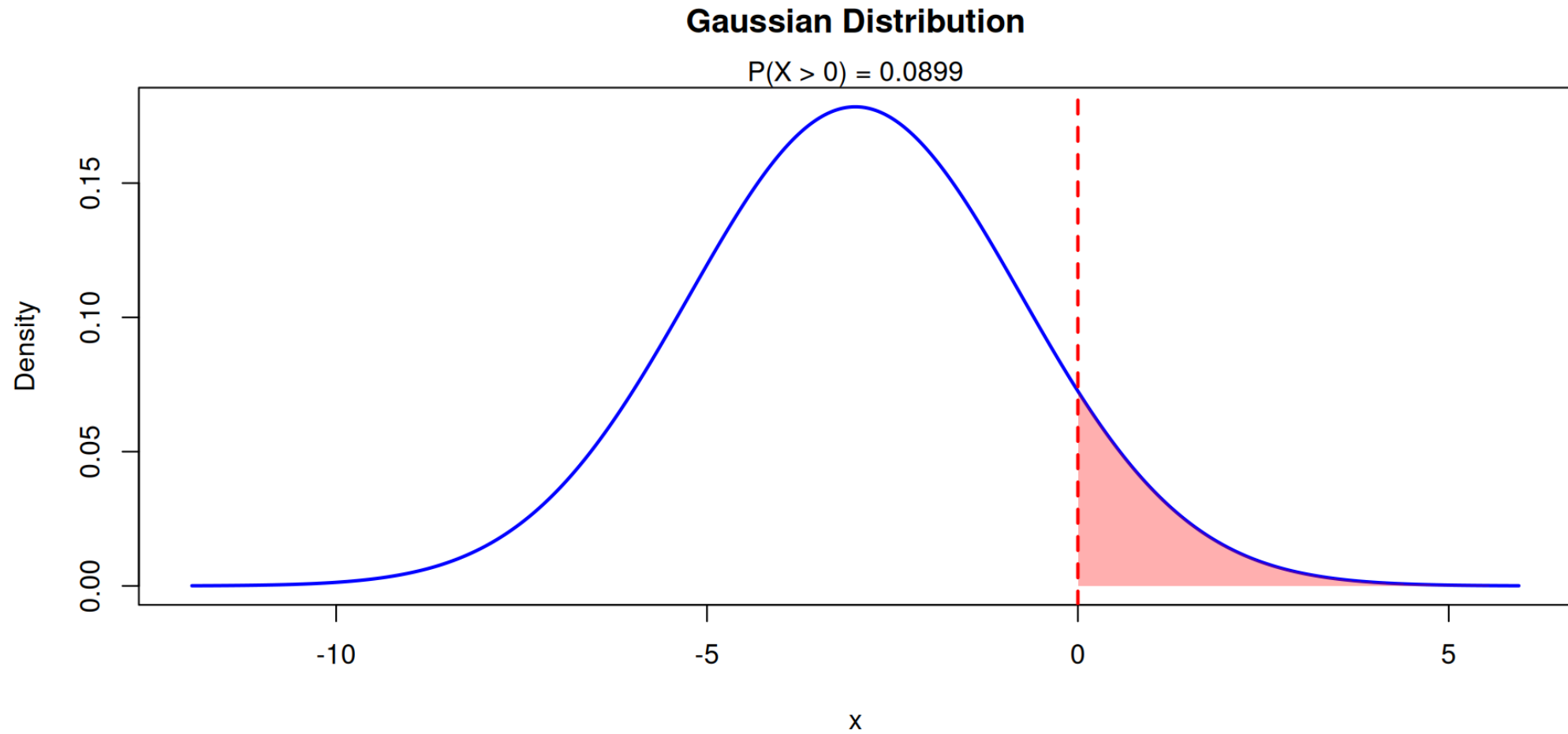
## Example 9

Find  $P(T > 0)$

$$T \sim N(-3, 5)$$

$$\begin{aligned} P(T > 0) &= P\left(Z > \frac{0 + 3}{\sqrt{5}}\right) = 1 - P(Z \leq 1.34) = \\ &1 - \Phi(1.34) = 0.0901 \end{aligned}$$

# Example 9



# Normal Distribution

## ❗ Corollaries

1.  $X_i \sim N(\mu, \sigma^2)$  for  $i = 1, \dots, n$ , i.i.d.
2.  $T = X_1 + \dots + X_n$  with  $\bar{X} = \frac{T}{n}$

Then:

1.  $T \sim N(n \times \mu, n\sigma^2)$
2.  $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

## Example 10

Let  $X_i \sim N(120, 64)$  be r.v.s representing the number of bank deposits made in a specific day. Then  $T = X_1 + \cdots + X_5$  are the weekly deposits.

Find the probability of the weekly deposits exceed 620.

$T \sim N(600, 320)$  because of the Normal additivity property.

## Example 10

$$\begin{aligned} P(T > 620) &= P\left(Z > \frac{620 - 5 \times 120}{\sqrt{5} \times 8}\right) = \\ &1 - P(Z \leq 1.12) = 1 - \Phi(1.12) = 0.1314 \end{aligned}$$

## ? Normal · Question 1

If  $X \sim N(\mu, \sigma^2)$ , then  $Z = \frac{X - \mu}{\sigma}$  follows:

A.  $N(\mu, \sigma^2)$

B. an Exponential

C. a Uniform

D.  $N(0, 1)$

☒ **D.** Standardizing centres the variable at 0 and rescales it to unit variance. That is why one table is enough for every Normal.

## ? Normal · Question 2

For the standard Normal,  $\Phi(-z)$  equals:

A.  $1 - \Phi(z)$

B.  $\Phi(z)$

C.  $-\Phi(z)$

D.  $2\Phi(z)$

✓ **A.** The standard Normal is symmetric about 0, so the area to the left of  $-z$  equals the area to the right of  $z$ .



# Normal · Question 3

Let  $X \sim N(6, 25)$ .

- a) Compute  $P(X \leq 14)$  and  $P(X > 2)$ .
- b) Find  $k$  such that  $P(X \leq k) = 0.90$ .
- c) Let  $Y \sim N(4, 9)$  be independent of  $X$ . Compute  $P(X + Y > 15)$ .



## Normal · Solution

Here  $\mu = 6$  and  $\sigma = \sqrt{25} = 5$ .

$$\text{a) } P(X \leq 14) = \Phi\left(\frac{14-6}{5}\right) = \Phi(1.6) = 0.9452$$

$$P(X > 2) = 1 - \Phi\left(\frac{2-6}{5}\right) = 1 - \Phi(-0.8) = \Phi(0.8) = 0.7881$$

**b)**  $\Phi(z) = 0.90$  gives  $z \approx 1.28$  from the table (1.282 interpolated), so  $k = 6 + 1.282 \times 5 \approx 12.41$

**c)** By additivity,  $X + Y \sim N(6 + 4, 25 + 9) = N(10, 34)$ , so

$$P(X + Y > 15) = 1 - \Phi\left(\frac{15 - 10}{\sqrt{34}}\right) = 1 - \Phi(0.86) = 1 - 0.8051 = 0.1949$$

# Practice

# Practice

The exercises go from easy to hard:

-  warm-up: one formula, one step
-  standard: two or three steps
-  exam level: this is what you will find in the midterm
-  stretch: you need to combine several things

After these, a mock test with Topic 3 problems, like in the midterm.

## Practice · Exercise 1

Let  $X \sim N(12, 4)$ , and consider  $P(a \leq X \leq 15) = 0.7332$ . Find  $a$ .

$$P(X \leq 15) = \Phi\left(\frac{15-12}{2}\right) = \Phi(1.5) = 0.9332$$

So  $\Phi\left(\frac{a-12}{2}\right) = 0.9332 - 0.7332 = 0.2$ , and from the table  $\frac{a-12}{2} \approx -0.84$

$$a \approx 12 - 2 \times 0.84 = 10.32$$

## Practice · Exercise 2

The weight of a package , in grams, is  $X \sim N(250, 100)$ .

- (a)** Find the probability that a package weighs between 230.4 and 269.6 grams.
- (b)** Packages farther away than 15.3 grams from the mean are rejected. Find the probability that a package is rejected.
- (c)** Find  $k$  such that  $P(X < k) = 0.9$ , and interpret it.
- (d)** Pick 4 packages. Find the probability that the total weight is above its mean.



## Practice · Exercise 2 · Solution

$\sigma = 10$ . **(a)**  $P(|Z| < \frac{19.6}{10}) = P(|Z| < 1.96) = 0.95$

**(b)**  $P(|Z| > 1.53) = 2(1 - \Phi(1.53)) = 2(1 - 0.9370) = 0.126$

**(c)**  $\Phi(z) = 0.9 \Rightarrow z \approx 1.28$ , so  $k = 250 + 1.28 \times 10 = 262.8$  grams. 90% of the packages weigh less than 262.8 grams.

**(d)** The total is Normal, and a Normal has half of its probability above its mean. So 0.5, no computation needed 😎

## Practice · Exercise 3

Let  $X \sim N(1, 1)$  and  $Z \sim N(0, 1)$  be independent.

**(a)** Find  $P(X < Z + 1)$ .

**(b)** Find  $a$  such that  $P(X > Z + a) = 0.1$ .

Put everything on one side:  $D = X - Z \sim N(1 - 0, 1 + 1) = N(1, 2)$ .

The variances **add**, even if we subtract.

**(a)**  $P(D < 1) = 0.5$ , since 1 is the mean of  $D$ .

**(b)**  $P(D > a) = 0.1 \Rightarrow \frac{a-1}{\sqrt{2}} \approx 1.282 \Rightarrow a \approx 1 + 1.282 \times 1.414 \approx 2.813$

## Practice · Exercise 4

True or false?

**(a)** If  $X \sim N(0, 1)$  and  $Y \sim N(1, 2)$  are independent, then  $2X + Y \sim N(1, 6)$ .

**(b)**  $\Phi(-x) = \Phi(x)$  for all  $x \in \mathbb{R}$ .

**(c)**  $\phi(-x) = 1 - \phi(x)$  for all  $x \in \mathbb{R}$ .

**(a)**   $E = 2(0) + 1 = 1$  and  $V = 4(1) + 2 = 6$

**(b)**   $\Phi(-x) = 1 - \Phi(x)$ . The areas in the tails are symmetric, not the cumulative probabilities.

**(c)**  The density is symmetric,  $\phi(-x) = \phi(x)$ . You mixed up (b) and (c)



# Mock Test

# Mock Test

Three problems, with the style and the length of the midterm. Try them in **40 minutes**, with the tables and a calculator, and no slides.

# Mock Test · Problem 1

A call center  receives, on average, 3 calls every 10 minutes, following a Poisson process.

- (a)** Find the probability of no calls in 10 minutes.
- (b)** Find the probability of at least 2 calls in 5 minutes.
- (c)** Find the expected waiting time until the next call, and the probability of waiting more than 5 minutes.



## Mock Test · Problem 1 · Solution

(a)  $X \sim Poi(3)$ :  $P(X = 0) = F(0) = 0.0498$

(b) In 5 minutes,  $Y \sim Poi(1.5)$ :

$$P(Y \geq 2) = 1 - F(1) = 1 - 0.5578 = 0.4422$$

(c) The time between calls, in minutes, is  $T \sim Exp(0.3)$ , so

$$E[T] = \frac{10}{3} \approx 3.33 \text{ minutes.}$$

$P(T > 5) = e^{-1.5} \approx 0.2231$ , which is exactly  $P(Y = 0)$  from (b): waiting more than 5 minutes is the same as no calls in 5 minutes.

## Mock Test · Problem 2

The weight of a bag of coffee , in grams, is  $X \sim N(500, 16)$ . Bags weighing less than 494.88 grams are rejected.

- (a)** Find the probability that a bag is rejected.
- (b)** A box has 10 bags, independent. Find the probability that at most one bag in the box is rejected.
- (c)** Find the probability that a box of 10 bags weighs less than 4980 grams.
- (d)** Find  $k$  such that 95% of the bags weigh more than  $k$ .



## Mock Test · Problem 2 · Solution

**(a)**  $P(X < 494.88) = \Phi\left(\frac{494.88-500}{4}\right) = \Phi(-1.28) = 1 - 0.8997 \approx 0.10$

**(b)**  $N \sim \text{Bin}(10, 0.1)$  counts the rejected bags:  $P(N \leq 1) = 0.7361$

**(c)**  $T = X_1 + \cdots + X_{10} \sim N(5000, 160)$ :  $P(T < 4980) = \Phi\left(\frac{-20}{\sqrt{160}}\right) = \Phi(-1.58) = 1 - 0.9429 = 0.0571$

**(d)**  $P(X > k) = 0.95 \Rightarrow \frac{k-500}{4} = -1.645 \Rightarrow k \approx 493.42$  grams.

## Mock Test · Problem 3

In a factory , 4% of the parts are defective. Parts are inspected one by one, independently.

**(a)** How many parts are expected to be inspected until the first defective one?

**(b)** Find the probability that the first defective part appears after the 10th inspection.

**(c)** A batch of 100 parts contains exactly 4 defective ones. You take 5 without replacement. Find the probability that none is defective, exactly and with the Binomial approximation.



## Mock Test · Problem 3 · Solution

(a)  $X \sim \text{Geo}(0.04)$ ,  $E[X] = \frac{1}{0.04} = 25$  parts.

(b)  $P(X > 10) = 0.96^{10} \approx 0.6648$

(c)  $Y \sim \text{Hypergeometric}(100, 4, 5)$ :

$$P(Y = 0) = \frac{\binom{4}{0} \binom{96}{5}}{\binom{100}{5}} \approx 0.8119$$

$\frac{n}{N} = 0.05 < 0.1$ , so  $Y \approx \text{Bin}(5, 0.04)$  and  $P(Y = 0) \approx 0.96^5 \approx 0.8154$ . Very close.



# Homework

Problem set **3.2**, Questions 8 to 13.

Good luck in the midterm 🍀

# References

- Murteira, B.; Silva Ribeiro, C.; Andrade e Silva, J. & Pimenta, C., Introdução à Estatística, Escolar Editora, McGraw-Hill, 2010
- Paulino, C. D. & Branco, J. A. (2005). Exercícios de Probabilidade e Estatística. Escolar Editora
- Pimenta, F., Andrade e Silva, J.; Silva Ribeiro, C. & Murteira, B., Introdução à Estatística, 3ª Edição, Escolar Editora, 2015
- Ferreira, T., Custódio, S.G., Modelos Probabilísticos, Síntese Teórica e Exercícios Resolvidos, Edições Sílabo (1ª Edição), 2023

# Appendix

# Standard Normal Table

| z   | .00    | .01    | .02    | .03    | .04    | .05    | .06    | .07    | .08    | .09    |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
|     |        |        |        |        |        |        |        |        |        |        |